STABILITY OF FOLIATIONS INDUCED BY RATIONAL MAPS

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Abstract. We show that the singular holomorphic foliations induced by dominant quasi-homogeneous rational maps fill out irreducible components of the space $\mathcal{F}_q(r,d)$ of singular foliations of codimension q and degree d on the complex projective space $\mathbb{P}^r$, when $1 \le q \le r-2$. We study the geometry of these irreducible components. In particular we prove that they are all rational varieties and we compute their projective degrees in several cases.

1. Introduction

1.1. **The space of codimension one holomorphic foliations on $\mathbb{P}^r$.** Let us consider a differential 1-form in $\mathbb{C}^{r+1}$

$$\omega = \sum_{i=0}^{r} a_i dx_i$$

where the a_i are homogeneous polynomials of degree $d+1$ in variables $x_0, \ldots, x_r$, with complex coefficients. Assume that $\sum_{i=0}^r a_i x_i = 0$, so that ω descends to the complex projective space $\mathbb{P}^r$ and defines a global section of the twisted sheaf of 1-forms $\Omega^1_{\mathbb{P}^r}(d+2)$.

The space of codimension one foliations of degree d on $\mathbb{P}^r$ is the algebraic subset of $\mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r}(d+2))\right)$ consisting of the 1-forms ω that satisfy the Frobenius integrability condition and has zero set of codimension at least two, i.e.,

$$\mathcal{F}(r,d) = \left\{ \omega \in \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r}(d+2))\right) \mid \omega \wedge d\omega = 0 \text{ and } \operatorname{codim} \operatorname{sing}(\omega) \ge 2 \right\}.$$

For the study of the irreducible components of $\mathcal{F}(r,d)$ we refer to e. g. [2] and [10].

1.2. **Stability of quasi-homogeneous pencils.** One of the first results on the subject is due to Gómez-Mont and Lins Neto [7] who proved that there are irreducible components $\mathcal{R}(r,d,d) \subset \mathcal{F}(r, 2d-2)$, $r \ge 3$, whose generic element is a foliation tangent to a Lefschetz pencil of degree d hypersurfaces. Their proof explores the topology of the underlying real foliation and relies on the stability of the Kupka components of the singular set and on Reeb's Leaf Stability Theorem. Using similar methods they recognized for $r \ge 4$ other irreducible components $\mathcal{R}(r, d_0, d_1) \subset \mathcal{F}(r, d_0+d_1-2)$ with generic member tangent to a quasi-homogeneous pencil $\langle \lambda F^{p_0} - \mu G^{p_1} \rangle$ with p_0 and p_1 relatively prime natural numbers satisfying $p_0 d_0 = p_1 d_1$, $d_i = \deg F_i$. Later Calvo-Andrade [1] extended Gómez-Mont-Lins Neto result about quasi-homogeneous pencils to dimension three. His proof has an extra dynamical ingredient –the stability of leaves carrying non-trivial holonomy.

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In fact in both of the above mentioned papers the authors do not restrict to $\mathbb{P}^r$ and prove their results for foliations on an arbitrary projective manifold M with $\dim M \geq 3$ and $\mathrm{H}^1(M, \mathbb{C}) = 0$. An alternative proof of the above results based on extension techniques of transversely euclidean structures has been carried out by Scárdua in [15].

1.3. **Infinitesimal stability of quasi-homogeneous pencils.** Although full of geometric insights the above mentioned works do not seem to shed any light on the scheme structure or the geometry of $\mathcal{R}(r, d_0, d_1)$. The present article stems from an attempt to understand these problems.

Using infinitesimal techniques, as in [4], we describe the Zariski tangent space of $\mathcal{R}(r, d_0, d_1)$ at a generic point and arrive at a proof that $\mathcal{R}(r, d_0, d_1)$ –with the natural scheme structure given by the Frobenius integrability condition– is *generically reduced*. More precisely if $\mathcal{R}(r, d_0, d_1)$ denotes the closure of the image of the rational map

$$\begin{array}{rcl} \rho : \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d_0))\right) \times \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d_1))\right) & \dashrightarrow & \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^1(d_0 + d_1))\right) \\ (F_0, F_1) & \longmapsto & d_0 F_0 dF_1 - d_1 F_1 dF_0 . \end{array}$$

then our first result reads as follows.

Theorem 1. *If $r \geq 3$ then $\mathcal{R}(r, d_0, d_1)$ is an irreducible and* <u>*generically reduced*</u> *component of $\mathcal{F}(r, d_0 + d_1 - 2)$.*

As explained above the only novelty in Theorem 1, besides the method of its proof, is what concerns the scheme structure over a generic point. For a more precise statement see Theorem 2.1 in §2.

The main content of this article is the generalization of Theorem 1 to foliations of higher codimension.

1.4. **Foliations on $\mathbb{P}^r$ of higher codimension.** Let ω be a homogeneous q-form on $\mathbb{C}^{r+1}$ with coefficients of degree $d+1$ that is annihilated by Euler's vector field. As before ω can be interpreted as a section of the sheaf of twisted differential q-forms $\Omega^q_{\mathbb{P}^r}(d + q + 1)$.

We recall from [13] (see also [4]) that ω defines a degree d holomorphic foliation of codimension q on $\mathbb{P}^r$ if it satisfies both Plücker's decomposability condition

$$(i_v \omega) \wedge \omega = 0 \quad \text{for every } v \in \bigwedge^{q-1} \mathbb{C}^{r+1}, \tag{1}$$

and the integrability condition

$$(i_v \omega) \wedge d\omega = 0 \quad \text{for every } v \in \bigwedge^{q-1} \mathbb{C}^{r+1}. \tag{2}$$

It is therefore natural to set $\mathcal{F}_q(r, d)$, the space of codimension q holomorphic foliations of degree d on $\mathbb{P}^r$, as

$$\left\{ \omega \in \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^q_{\mathbb{P}^r}(d + q + 1))\right) \,\middle|\, \omega \text{ satisfies } (1), (2) \text{ and } \operatorname{codim} \operatorname{sing}(\omega) \geq 2 \right\}.$$

1.5. **Infinitesimal stability of quasi-homogeneous rational maps.** If one interprets the elements of $\mathcal{R}(r,d_0,d_1)$ as foliations tangent to the fibers of rational maps

$$\begin{array}{ccc} \mathbb{P}^r & \dashrightarrow & \mathbb{P}^1 \\ x & \mapsto & (F^{p_0} : G^{p_1}) \end{array}$$

then a possible counterpart in the higher codimension case are the foliations tangent to dominant rational maps $\mathbb{P}^r \dashrightarrow \mathbb{P}^q$.

When $q = r-1$ there is no hope to establish a stability result even for a generic rational map. Indeed, under this constraint both Plücker's condition and the integrability condition are vacuous. Thus $\mathcal{F}_{r-1}(r,d)$ can be identified with an open subset of $\mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r,\Omega^{r-1}_{\mathbb{P}^r}(d+r))\right) = \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, T\mathbb{P}^r(d-1))\right)$. It is well known that for $d \geq 2$ a generic element of this space has no algebraic leaves, see for instance [3].

For $1 \leq q \leq r-2$ fix integers $d_0,\ldots,d_q$ and consider homogeneous polynomials F_i of degree d_i for $i = 0,\ldots,q$. Assume that the q-form

$$\omega = i_R(dF_0 \wedge \cdots \wedge dF_q), \tag{3}$$

is non-zero. It is easy to check that ω satisfies both (1) and (2) since $i_v\omega = \sum a_{ij} i_R(dF_i \wedge dF_j)$, where the a_{ij} are homogeneous polynomials. Moreover, it defines a foliation tangent to the fibers of the map

$$\begin{array}{ccc} \mathbb{P}^r & \dashrightarrow & \mathbb{P}^q \\ x & \mapsto & (F_0^{e_0} : \ldots : F_q^{e_q}) \end{array}$$

with $e_i = \mathrm{lcm}(d_0,\ldots,d_q)/d_i$. We set

$$d = \sum d_i - q - 1$$

and denote by

$$\mathcal{R}(r,d_0,\ldots,d_q) \subset \mathcal{F}_q(r,d)$$

the closure of the set of foliations that can be written in the form (3). It is the closure of the image of the rational map

$$\begin{array}{rccc} \rho : \prod_i \mathbb{P}\left(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^r}(d_i))\right) & \dashrightarrow & \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r,\Omega^1(d+q+1))\right) \\ (F_i) & \mapsto & i_R(dF_0 \wedge \cdots \wedge dF_q). \end{array}$$

Notice that for $q = 1$ we recover the definition of $\mathcal{R}(r,d_0,d_1)$.

Theorem 2. *If $r \geq 4$ and $1 \leq q \leq r-2$ then $\mathcal{R}(r,d_0,\ldots,d_q)$ is an irreducible and generically reduced component of $\mathcal{F}_q\left(r, \sum d_i - q - 1\right)$.*

As far as we know there is no information in the literature concerning the geometry of the irreducible components of $\mathcal{F}_q(r,d)$ so far.

1.6. **Geometry of the rational components.** In Section 3 we initiate this study through an investigation of the parameterization ρ. Besides computing the dimension of $\mathcal{R}(r,d_0,\ldots,d_q)$, we prove the following.

Theorem 3. *The irreducible components $\mathcal{R}(r,d_0,\ldots,d_q)$ are rational varieties.*

By its definition, $\mathcal{R}(r, d_0, \ldots, d_q)$ is unirational. The proof of rationality relies on the construction of a variety X that sits as an open set in the total space of a tower of Grasmmann bundles, together with a birational morphism $p : X \to \mathcal{R}(r, d_0, \ldots, d_q)$.

In general we do not know how to naturally compactify X to a projective variety where p extends to a morphism. Albeit, in a number of cases we are able to do that and obtain, with the aid of Schubert Calculus, formulas for the degree of the projective subvarities

$$\mathcal{R}(r, d_0, \ldots, d_q) \subset \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^q(d+q+1))\right).$$

For example the first few values for the degree of $\mathcal{R}(r, 2, 2, 2)$ are listed below.

r	Degree
3	**1324220**
4	**2860923458080**
5	**2436619729804777736263**
6	**72844073370510783178951724585 8**
7	**7046130965135851235853984086962318991761 83**

Several other cases are treated in Section 5.

Contents

2. Infinitesimal stability of quasi-homogeneous pencils

In this first section we present our proof of Theorem 1. All the arguments will be reworked later in greater generality. We felt the exposition of this particular case of Theorem 2 would improve the clarity of the paper.

For simplicity, let us denote by

$$\mathbf{S}_e = \mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(e)) \tag{4}$$

the vector space of homogeneous polynomials of degree e in $r+1$ variables, and

$$\mathcal{F} = \mathcal{F}(r, d)$$

so that our rational map ρ is

$$\rho : \mathbb{P}(\mathbf{S}_{d_0}) \times \mathbb{P}(\mathbf{S}_{d_1}) \dashrightarrow \mathcal{F} \subset \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^1(d+2))\right). \tag{5}$$

If p_0 and p_1 denote the unique coprime natural numbers such that $p_0 d_0 = p_1 d_1$ then

$$\rho(F_0, F_1) = d_0 F_0 dF_1 - d_1 F_1 dF_0 = p_1 F_0 dF_1 - p_0 F_1 dF_0$$

where the last equality of differential forms is up to multiplicative constant.

We remark that

$$d\left(\frac{F_0^{p_0}}{F_1^{p_1}}\right) = \frac{F_0^{p_0-1}}{F_1^{p_1+1}}\,(p_1 F_0 dF_1 - p_0 F_1 dF_0).$$

Therefore, the closure of the leaves of the singular foliation defined by the integrable 1-form $\rho(F_0, F_1)$ are irreducible components of the members of the pencil of hypersurfaces of degree $p_0 d_0 = p_1 d_1$ generated by $F_0^{p_0}$ and $F_1^{p_1}$.

2.1. The Zariski tangent space of $\mathcal{F}$.

For a scheme X and a point $x \in X$ we denote by $T_x X$ the Zariski tangent space of X at x. If $\mathbb{P}(V)$ is the projective space associated to a $\mathbb{C}$-vector space V and denoting $\pi : V - \{0\} \to \mathbb{P}(V)$ the canonical projection, for each $v \in V$ we have a natural identification

$$T_{\pi(v)}\mathbb{P}(V) = V/(v)$$

where (v) denotes de one-dimensional subspace generated by v. With slight abuse of notations, the Zariski tangent space $T_\omega \mathcal{F}$ of $\mathcal{F}$ at a point ω is represented by the forms $\eta \in \mathrm{H}^0(\mathbb{P}^r, \Omega^1(d+2))/(\omega)$ such that

$$(\omega + \epsilon\eta) \wedge (d\omega + \epsilon d\eta) = 0 \mod \epsilon^2$$

that is, such that

$$\omega \wedge d\eta + \eta \wedge d\omega = 0 \quad \text{or, equivalently} \quad d\omega \wedge d\eta = 0,$$

where the equivalence is implied by the following variant of Euler's formula for homogeneous polynomials.

Lemma 2.1. *If η is a homogeneous q-form with degree d coefficients then*

$$i_R d\eta + d(i_R \eta) = (q + d)\eta$$

where R is the radial or Euler vector field and i_R denotes the interior product or contraction with R.

Proof. See [10, Lemme 1.2, pp. 3]. □

Therefore to determine $T_\omega \mathcal{F}$ is equivalent to solve $d\omega \wedge d\eta = 0$. Notice that in the situation under scrutiny $d\omega = (d_0 + d_1) dF_0 \wedge dF_1$. The first step towards the general η satisfying $d\omega \wedge d\eta = 0$ is given by Saito's generalization of DeRham's division Lemma. In Lemma 2.2 we state variants of both DeRham's and Saito's Lemmas fine tuned up for our purposes.

Lemma 2.2 ([14]). *Let $F_0, \ldots, F_q$ be homogeneous polynomial functions on $\mathbb{C}^{r+1}$ and let $\Theta \in \Omega^{q+1}(\mathbb{C}^{r+1})$ be the $(q+1)$-form given by*

$$\Theta = dF_0 \wedge \ldots \wedge dF_q \, .$$

(a) *Suppose that $q < r$ and* $\operatorname{codim} \operatorname{sing}(\Theta) \geq 2$. *If $\eta \in \Omega^1(\mathbb{C}^{r+1})$ is a homogeneous polynomial $1-$form such that $\Theta \wedge \eta = 0$ then there exist homogeneous polynomials $a_0, \ldots, a_q$ such that*

$$\eta = \sum_{i=0}^{q} a_i dF_i .$$

(b) *Suppose that $q < r-1$ and* $\operatorname{codim} \operatorname{sing}(\Theta) \geq 3$. *If $\eta \in \Omega^2(\mathbb{C}^{r+1})$ is a homogeneous polynomial $2-$form such that $\Theta \wedge \eta = 0$ then there exist homogeneous polynomial 1-forms $\alpha_0, \ldots, \alpha_q$ such that*

$$\eta = \sum_{i=0}^{q} \alpha_i \wedge dF_i .$$

Remark 2.1. The hypothesis $q < r$ in (a) and $q < r - 1$ in (b) are not really necessary. For instance in item (b) the singular set $\operatorname{sing}(\Theta)$ equals the locus where the $(q+1) \times (r+1)$ Jacobian matrix $(\partial F_i / \partial x_j)$ has rank $\leq q$. Hence $\operatorname{sing}(\Theta)$ is empty or has codimension at most $r + 1 - q$. When $q \geq r - 1$ it follows that $\operatorname{codim} \operatorname{sing}(\Theta) \geq 3$ implies that Θ has no singularities. We conclude that $F_0, \ldots, F_q$ are linearly independent linear forms and the conclusion trivially holds true in this case.

In face of Lemma 2.2 it is natural to define the open subset

$$\mathcal{U} = \{\omega \in \mathcal{R}(r, d_0, d_1) \mid \text{codim } \operatorname{sing}(d\omega) \geq 3 \text{ and codim } \operatorname{sing}(\omega) \geq 2\}. \tag{6}$$

The next result will imply the infinitesimal stability of quasi-homogeneous pencils corresponding to points of $\mathcal{U}$. It is a simple particular case of Proposition 3.1. The iteration argument in the proof is generalized in Lemma 4.2. We feel it is worthwhile to write it here for the sake of clarity.

Proposition 2.1. *Let $(F_0, F_1) \in \mathbb{P}(\mathbf{S}_{d_0}) \times \mathbb{P}(\mathbf{S}_{d_1})$ be such that $\rho(F_0, F_1) = \omega \in \mathcal{U}$. Then the derivative*

$$d\rho(F_0, F_1) : T_{(F_0, F_1)}(\mathbb{P}(\mathbf{S}_{d_0}) \times \mathbb{P}(\mathbf{S}_{d_1})) \to T_\omega \mathcal{F}$$

is surjective. In other words, ρ is a submersion over $\mathcal{U}$.

Proof. It is convenient to write

$$\rho(F_0, F_1) = d_0 F_0 dF_1 - d_1 F_1 dF_0 = i_R(dF_0 \wedge dF_1).$$

Then, the derivative of ρ at the point (F_0, F_1)

$$d\rho(F_0, F_1) : \mathbf{S}_{d_0}/(F_0) \times \mathbf{S}_{d_1}/(F_1) \to T_\omega \mathcal{F}$$

is calculated as

$$d\rho(F_0, F_1)(F_0', F_1') = i_R(dF_0' \wedge dF_1 + dF_0 \wedge dF_1').$$

Let $\eta \in \mathrm{H}^0(\mathbb{P}^r, \Omega^1(d+2))$ represent an element of $T_\omega\mathcal{F}$, that is, $d\omega \wedge d\eta = 0$. We shall prove that η belongs to the image of $d\rho(F_0, F_1)$, *i.e.*,

$$\eta = i_R(dF_0' \wedge dF_1 + dF_0 \wedge dF_1')$$

for some $F_0' \in \mathbf{S}_{d_0}$ and $F_1' \in \mathbf{S}_{d_1}$.

Since $d\omega = dF_0 \wedge dF_1$, applying the division Lemma 2.2 to $d\eta$ it follows that there exist homogeneous 1-forms α and β such that

$$d\eta = \alpha \wedge dF_0 + \beta \wedge dF_1.$$

Notice that $d\eta$ is a 2-form with coefficients homogeneous polynomials of degree $d = d_0 + d_1 - 2$. Hence the coefficients of α (resp. β) are homogeneous of degree $d_1 - 1$ (resp. $d_0 - 1$). Applying exterior derivative we find

$$d\alpha \wedge dF_0 + d\beta \wedge dF_1 = 0.$$

Multiplying by dF_1 we get $d\alpha \wedge dF_0 \wedge dF_1 = 0$. From lemma 2.2 applied to $d\alpha$ we deduce

$$d\alpha = \alpha' \wedge dF_0 + \alpha'' \wedge dF_1$$

where α' and α'' are 1-forms with coefficients homogeneous polynomials of respective degrees $d_1 - 2 - (d_0 - 1) = d_1 - d_0 - 1$ and $d_1 - 2 - (d_1 - 1) = -1$. Hence $\alpha'' = 0$. Similarly,

$$d\beta = \beta' \wedge dF_0 + \beta'' \wedge dF_1$$

where β' and β'' are 1-forms with coefficients homogeneous polynomials of respective degrees $d_0 - 2 - (d_0 - 1) = -1$ and $d_0 - 2 - (d_1 - 1) = d_0 - d_1 - 1$. Hence $\beta' = 0$.

Suppose that $d_0 = d_1$. By the considerations above regarding degrees, $\alpha' = \beta'' = 0$. Thus α and β are closed 1-forms. Therefore $\alpha = -dF_1'$ and $\beta = dF_0'$ where F_i' is some homogeneous polynomial of degree d_i. It follows that $d\eta = dF_0' \wedge dF_1 + dF_0 \wedge dF_1'$ and since $i_R(d\eta) = (d+1)\eta$ we obtain that η is a scalar multiple of $i_R(dF_0' \wedge dF_1 + dF_0 \wedge dF_1')$. Therefore the Proposition is proved in the case $d_0 = d_1$.

Now suppose $d_0 \neq d_1$, say $d_0 > d_1$. Then $d_1 - d_0 - 1 < 0$. Hence $d\alpha = 0$ and $d\beta = \beta'' \wedge dF_1$. Repeating the argument of the previous case we obtain a sequence of 1-forms β_i, $i \in \mathbb{N}$, such that

$$d\beta_i = \beta_{i+1} \wedge dF_1$$

Comparing degrees it follows that, for $k \gg 0$, $\beta_k = 0$. Thus $d\beta_{k-1} = 0$ and there exists a homogeneous polynomial b_{k-1} such that $\beta_{k-1} = db_{k-1}$. Then $d\beta_{k-2} = db_{k-1} \wedge dF_1$ and hence $\beta_{k-2} = b_{k-1}dF_1 + db_{k-2}$ for a suitable homogeneous polynomial b_{k-2}. Then $d\beta_{k-3} = \beta_{k-2} \wedge dF_1 = db_{k-2} \wedge dF_1$. Hence there exists b_{k-3} such that $\beta_{k-3} = b_{k-2}dF_1 + db_{k-3}$. Iterating this, we conclude that $\beta = \beta_0 = b_1 dF_1 + db_0$ and therefore

$$d\eta = dF_1' \wedge dF_0 + dF_0' \wedge dF_1$$

where $dF_1' = \alpha$ and $dF_0' = db_0$, as wanted. □

2.2. **Proof of Theorem 1.** As a matter of fact we prove the following slightly more precise statement.

Theorem 2.1. *If $r \geq 3$ then $\mathcal{R}(r, d_0, d_1)$ is an irreducible component of $\mathcal{F}(r, d)$. Moreover, $\mathcal{F}(r, d)$ is smooth and reduced at the points of $\mathcal{U}$.*

Proof. Write as before $\rho : P \dashrightarrow \mathcal{F}$, where $P = \mathbb{P}(\mathbf{S}_{d_0}) \times \mathbb{P}(\mathbf{S}_{d_1})$, $\mathcal{F} = \mathcal{F}(r, d)$ and $\mathcal{R} = \mathcal{R}(r, d_0, d_1)$ is the closure of the image of ρ. Put $F = (F_0, F_1) \in P$. Proposition 2.1 implies that for $\omega = \rho(F)$, the derivative

$$d\rho(F) : T_F P \to T\mathcal{F}_\omega$$

is surjective and also factors through $T_\omega \mathcal{R} \subseteq T_\omega \mathcal{F}$. Then $T_\omega \mathcal{R} = T_\omega \mathcal{F}$. It follows that $\mathcal{R}$ is an irreducible component of $\mathcal{F}$ and $\mathcal{F}$ is reduced at the generic point of $\mathcal{R}$. □

3. Stability of quasi-homogeneous rational maps

In this section we exhibit some previously unknown irreducible components $\mathcal{R}(r, d_0, \ldots, d_q)$ of $\mathcal{F}_q(r, d)$, generalizing the case $q = 1$ of the previous section.

A point of $\mathcal{R}(r, d_0, \ldots, d_q)$ will be a twisted q-form $\omega \in \mathrm{H}^0(\mathbb{P}^r, \Omega^q(d + q + 1))$ of type

$$\omega = i_R(dF_0 \wedge \cdots \wedge dF_q) = \sum_{0 \leq j \leq q} (-1)^j d_j F_j \; dF_0 \wedge \cdots \wedge \widehat{dF_j} \wedge \cdots \wedge dF_q \tag{7}$$

where $F_j \in \mathbf{S}_{d_j}$ is a homogeneous polynomial of degree d_j in $r + 1$ variables, and

$$d_0 + \cdots + d_q = d + q + 1. \tag{8}$$

We call ω a *rational q-form* in $\mathbb{P}^r$ of type $(d_0, \ldots, d_q)$.

More precisely, $\mathcal{R}(r, d_0, \ldots, d_q)$ is defined as the closure of the image of the rational map

$$\rho : \mathbb{P}(\mathbf{S}_{d_0}) \times \cdots \times \mathbb{P}\left(\mathbf{S}_{d_q}\right) \dashrightarrow \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^q(d + q + 1))\right) \tag{9}$$

induced by the multilinear map

$$\mu : \mathbf{S}_{d_0} \times \cdots \times \mathbf{S}_{d_q} \to \mathrm{H}^0(\mathbb{P}^r, \Omega^q(d + q + 1))$$

such that $\mu(F_0, \ldots, F_q) = i_R(dF_0 \wedge \cdots \wedge dF_q)$. The base locus of ρ is described in (16) below.

As in the previous section, we define the open subset

$$\mathcal{U} = \{\omega \in \mathcal{R}(r, d_0, \ldots, d_q) \mid \operatorname{codim} \operatorname{sing}(d\omega) \geq 3 \text{ and } \operatorname{codim} \operatorname{sing}(\omega) \geq 2\}. \tag{10}$$

With notation as above, our main purpose in this section is to prove the following Theorem 3.1, which is a more precise version of Theorem 2 of the Introduction.

Theorem 3.1. *Suppose $r \geq 3$ and $1 \leq q \leq r - 2$. Then $\mathcal{R}(r, d_0, \ldots, d_q)$ is an irreducible component of $\mathcal{F}_q(r, d)$. Moreover, $\mathcal{F}_q(r, d)$ is smooth and reduced at the points of $\mathcal{U}$.*

The strategy is the same as the one used to prove Theorem 2.1. Let us denote by $\mathcal{F} = \mathcal{F}_q(r, d)$. The scheme $\mathcal{F}$ is defined by the quadratic equations

$$i(v_J)\omega \wedge \omega = 0 \quad \text{and} \quad i(v_J)\omega \wedge d\omega = 0 \tag{11}$$

for all $J \subset \{0, \ldots, r\}$ of cardinality $q - 1$.

The tangent space $T_\omega \mathcal{F}$ of $\mathcal{F}$ at a point ω is represented by the forms $\omega' \in \mathrm{H}^0(\mathbb{P}^r, \Omega^q(d+q+1))/(\omega)$ such that $\omega_\epsilon = \omega + \epsilon\omega'$ satisfies the conditions (11) modulo ϵ^2, that is

$$i(v_J)\omega_\epsilon \wedge \omega_\epsilon = 0 \quad \text{and} \quad i(v_J)\omega_\epsilon \wedge d\omega_\epsilon = 0$$

modulo ϵ^2, for all $J \subset \{0, \dots, r\}$ of cardinality $q-1$. Expanding, one obtains

$$i(v_J)\omega' \wedge \omega + i(v_J)\omega \wedge \omega' = 0 \quad \text{and} \quad i(v_J)\omega' \wedge d\omega + i(v_J)\omega \wedge d\omega' = 0. \tag{12}$$

In order to work out ω' from (12) we will need a pair of technical results.

3.1. **Lemmata.** The first technical Lemma is a generalization of Lemma 2.2 that will be a central tool in the rest of this article.

Lemma 3.1. *l Let $F_0, \dots, F_q$ be homogeneous polynomial functions on $\mathbb{C}^{r+1}$ and let $\Theta \in \Omega^{q+1}(\mathbb{C}^{r+1})$ be the $(q+1)$-form given by*

$$\Theta = dF_0 \wedge \dots \wedge dF_q\,.$$

Suppose that $\operatorname{codim} \operatorname{sing}(\Theta) \geq 3$. *If $\eta \in \Omega^{q+1}(\mathbb{C}^{r+1})$ is such that $\eta \wedge dF_i \wedge dF_j = 0$ for every $0 \leq i < j \leq q$ then there exist holomorphic 1-forms $\alpha_0, \dots, \alpha_q \in \Omega^1(\mathbb{C}^{r+1})$ such that*

$$\eta = \sum_{i=0}^{q} \alpha_i \wedge dF_0 \wedge \dots \widehat{dF_i} \dots \wedge dF_q.$$

Proof. For the second item let $\mathcal{U}$ be an open covering of $\mathbb{C}^{r+1} \setminus \operatorname{sing}(\Theta)$. Since $\operatorname{codim} \operatorname{sing}(\Theta) \geq 3$ we can assume that over each open set $U \in \mathcal{U}$ our set of functions is part of a coordinate system on U. It is then clear that

$$\eta_{|U} = \sum \alpha_{i,U} \wedge dF_0 \wedge \dots \widehat{dF_i} \dots \wedge dF_q$$

for suitable 1-forms $\alpha_{0,U}, \dots, \alpha_{q,U} \in \Omega^1(U)$.

A simple computation shows that over $U \cap V$

$$(\alpha_{i,U} - \alpha_{i,V}) \wedge \Theta = 0\,.$$

It follows from Saito's Lemma [14] that there exists a unique $(q+1)\times(q+1)$ matrix $A_{U\cap V}$ with entries in $\mathcal{O}(U \cap V)$ such that

$$\begin{bmatrix} \alpha_{0,U} - \alpha_{0,V} \\ \vdots \\ \alpha_{q,U} - \alpha_{q,V} \end{bmatrix} = A_{U\cap V} \cdot \begin{bmatrix} dF_0 \\ \vdots \\ dF_q \end{bmatrix}$$

Of course the collection of matrices $A_{U\cap V}$ with (U, V) ranging in $\mathcal{U}^2$ defines an element of $\mathrm{H}^1(\mathbb{C}^{r+1} \setminus \operatorname{sing}(\Theta), \mathbb{M} \otimes \mathcal{O}) \cong \mathrm{H}^1(\mathbb{C}^{r+1} \setminus \operatorname{sing}(\Theta), \mathcal{O}) \otimes \mathbb{M}$, with $\mathbb{M}$ being the vector space of $(q+1) \times (q+1)$ matrices.

The hypothesis $\operatorname{codim} \operatorname{sing}(\Theta) \geq 3$ implies that this cohomology group is trivial, see for instance [6, pg. 133]. Therefore we may write $A_{U\cap V} = A_U - A_V$ where A_U, A_V are matrices of holomorphic functions in U resp. V. We can thus set

$$\begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_q \end{bmatrix} = \begin{bmatrix} \alpha_{0,U} \\ \vdots \\ \alpha_{q,U} \end{bmatrix} - A_U \cdot \begin{bmatrix} dF_0 \\ \vdots \\ dF_q \end{bmatrix} = \begin{bmatrix} \alpha_{0,V} \\ \vdots \\ \alpha_{q,V} \end{bmatrix} - A_V \cdot \begin{bmatrix} dF_0 \\ \vdots \\ dF_q \end{bmatrix}$$

as the sought global 1-forms at least over $\mathbb{C}^{r+1} \setminus \mathrm{sing}(\Theta)$. To conclude one has just to invoke Hartog's extension Theorem to ensure that these 1-forms extend to $\mathbb{C}^{r+1}$. $\square$

By expanding in its homogeneous components both sides of the equality

$$\eta = \sum_{i=0}^{q} \alpha_i \wedge dF_0 \wedge \ldots \widehat{dF_i} \ldots \wedge dF_q.$$

it can be easily seen that if η is a homogeneous polynomial q-form then the 1-forms $\alpha_0, \ldots, \alpha_q$ can be assumed homogeneous polynomial 1-forms.

The second technical Lemma in this subsection replaces the iteration argument in the proof of Theorem 2.1

Lemma 3.2. *For $j = 0, \ldots, q$ let $F_j \in \mathbf{S}_{d_j}$ be a homogeneous polynomial of degree d_j. Suppose $\omega = i_R(dF_0 \wedge \cdots \wedge dF_q)$ satisfies* codim sing $(d\omega) \geq 3$. *Then, for $\alpha \in \mathrm{H}^0(\mathbb{P}^r, \Omega^1(e))$ the following conditions are equivalent:*

(a) $d\alpha = \sum_{0 \leq k \leq q} A_k \wedge dF_k$ *for some* $A_k \in \mathrm{H}^0(\mathbb{P}^r, \Omega^1(e - d_k))$.
(b) $\alpha = dG + \sum_{0 \leq k \leq q} H_k \, dF_k$ *for some* $G \in \mathbf{S}_e$ *and* $H_k \in \mathbf{S}_{e-d_k}$.

Proof. It is clear that (b) implies (a). Let us prove the converse, by induction on $e \in \mathbb{N}$. If (a) holds, applying exterior derivative we get

$$0 = d^2\alpha = \sum_{0 \leq k \leq q} dA_k \wedge dF_k \implies dA_k \wedge dF_0 \wedge \cdots \wedge dF_q = 0.$$

By the hypothesis on the F_j and Lemma 2.2,

$$dA_k = \sum_{0 \leq h \leq q} A_{kh} \wedge dF_h$$

for some $A_{kh} \in \mathrm{H}^0(\mathbb{P}^r, \Omega^1(e - d_k - d_h))$. Since $e - d_k < e$, the inductive hypothesis applies to A_k and yields

$$A_k = dG_k + \sum_{0 \leq h \leq q} H_{kh} \, dF_h$$

for some $G_k \in \mathbf{S}_{e-d_k}$ and $H_k \in \mathbf{S}_{e-d_k-d_h}$. Replacing in (a) we find

$$d\alpha = \sum_k dG_k \wedge dF_k + \sum_{h,k} H_{kh} \, dF_h \wedge dF_k.$$

Since $i_R\alpha = 0$, we have $e{\cdot}\alpha = i_R d\alpha$. Applying i_R we obtain, after a little calculation

$$e \cdot \alpha = dG + \sum_{0 \leq k \leq q} H_k \, dF_k$$

where

$$G = -\sum_k d_k F_k G_k, \quad H_k = (d_k + e)G_k + \sum_h d_h F_h (H_{kh} - H_{hk})$$

as claimed. $\square$

3.2. **Surjectivity of the derivative and proof of Theorem 2.** Now we are ready to complete the proof of Theorem 3.1 and hence of Theorem 2 of the Introduction. The proof follows from Proposition 3.1 below combined with the same argument used in the proof of Theorem 2.1.

Proposition 3.1. *Suppose $r \geq 3$ and $1 \leq q < r-1$. If $\underline{F} = (F_0, \dots, F_q) \in \prod_i \mathbb{P}(\mathbf{S}_{d_i})$ is such that $\rho(\underline{F}) = \omega \in \mathcal{U}$ then the derivative*

$$d\rho(\underline{F}) : T_{\underline{F}}(\mathbb{P}(\mathbf{S}_{d_0}) \times \cdots \times \mathbb{P}\left(\mathbf{S}_{d_q}\right)) \to T_\omega \mathcal{F}$$

is surjective.

Proof. At a point $\underline{F} = (F_0, \dots, F_q)$ belonging to the domain of ρ the derivative

$$d\rho(\underline{F}) : \mathbf{S}_{d_0}/(F_0) \times \cdots \times \mathbf{S}_{d_q}/(F_q) \to T_\omega \mathcal{F} \tag{13}$$

is calculated by multilinearity as

$$d\rho(\underline{F})(F_0', \dots, F_q') = \sum_{0 \leq j \leq q} i_R(dF_0 \wedge \cdots \wedge dF_j' \wedge \cdots \wedge dF_q).$$

Let $\omega = \rho(\underline{F}) \in \mathcal{U}$ and $\omega' \in T_\omega \mathcal{F}$. From (12) we have

$$i(v_J)\omega' \wedge d\omega = -i(v_J)\omega \wedge d\omega'.$$

Since $d\omega$ is a constant multiple of $dF_0 \wedge \cdots \wedge dF_q$ (see Lemma 2.1), by exterior multiplication with dF_j we obtain

$$dF_j \wedge i(v_J)\omega \wedge d\omega' = 0$$

for all j, J.

Let $Y_j, (0 \leq j \leq q)$, be rational vector fields such that $dF_i(Y_j) = \delta_{ij}$. For $J = \{0, \dots, q\} \setminus \{i, j\}$ we have $i(v_J)\omega = \lambda(F_i dF_j - F_j dF_i)$. Then,

$$0 = dF_j \wedge i(v_J)\omega \wedge d\omega' = \lambda dF_j \wedge F_j dF_i \wedge d\omega',$$

which implies that

$$dF_i \wedge dF_j \wedge d\omega' = 0$$

for all $0 \leq i, j \leq q$.

Lemma 3.1 implies that

$$d\omega' = \sum_{0 \leq j \leq q} \alpha_j \wedge dF_0 \wedge \cdots \wedge \widehat{dF_j} \wedge \cdots \wedge dF_q \tag{14}$$

for some $\alpha_j \in \mathrm{H}^0(\mathbb{P}^r, \Omega^1(d_j))$. Applying exterior derivative we find

$$0 = d^2\omega' = \sum_{0 \leq j \leq q} d\alpha_j \wedge dF_0 \wedge \cdots \wedge \widehat{dF_j} \wedge \cdots \wedge dF_q.$$

Taking wedge product with dF_j we get

$$d\alpha_j \wedge (dF_0 \wedge \cdots \wedge dF_q) = 0$$

for all j. Therefore, thanks to Lemma 2.2,

$$d\alpha_j = \sum_{0 \leq k \leq q} A_{jk} \wedge dF_k$$

for suitable $A_{jk} \in \mathrm{H}^0(\mathbb{P}^r, \Omega^1(d_j - d_k))$. Lemma 3.2 implies that

$$\alpha_j = dG_j + \sum_{0 \leq k \leq q} H_{jk}\ dF_k$$

for some $G_j \in \mathbf{S}_{d_j}$ and $H_{jk} \in \mathbf{S}_{d_j - d_k}$ (we use the convention $\mathbf{S}_e = 0$ for $e < 0$). Replacing in (14) above we have

$$d\omega' = \sum_{0 \le j \le q} dG_j \wedge dF_0 \wedge \cdots \wedge \widehat{dF_j} \wedge \cdots \wedge dF_q + c \; dF_0 \wedge \cdots \wedge dF_q \tag{15}$$

for some $c \in \mathbb{C}$. Since $i_R \omega' = 0$, Lemma 2.1 yields $(\sum_i d_i)\ \omega' = i_R d\omega'$. Applying i_R to (15) and taking (13) into account, we obtain

$$\omega' = d\rho(\underline{F})(F'_0, \ldots, F'_q)$$

where $F'_j = \frac{(-1)^j}{(\sum_i d_i)}\ G_j$. Therefore $d\rho(\underline{F})$ is surjective, as claimed. □

4. Geometry of the parametrization

In this section we analyze the parametrization

$$\rho : \mathbb{P}\left(\mathbf{S}_{d_0}\right) \times \cdots \times \mathbb{P}\left(\mathbf{S}_{d_q}\right) \dashrightarrow \mathcal{R}_q(r, \bar{d}) \subset \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^q(d+q+1))\right) ,$$

where $\mathbf{S}_{d_i} = \mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d_i))$, $d = \sum d_i$ and $\bar{d} = (d_0, \ldots, d_q)$.

4.1. **Base locus.** Let us start by describing the base locus $\mathbf{B}(\rho)$ of ρ.

If $i_R(dF_0 \wedge \cdots \wedge dF_q) = 0$, applying exterior differentiation and Lemma 2.1 we obtain that $dF_0 \wedge \cdots \wedge dF_q = 0$. This means that the Jacobian matrix of $F_0, \ldots, F_q$ has rank $< q+1$ everywhere, that is, the derivative of the map

$$\underline{F} : \mathbb{C}^{r+1} \to \mathbb{C}^{q+1}$$

defined by $\underline{F}(x) = (F_0(x), \ldots, F_q(x))$ has rank $< q+1$ at every $x \in \mathbb{C}^{r+1}$. This is equivalent to the fact that F is not dominant, that is, $f(F_0, \ldots, F_q) = 0$ for some non-zero polynomial $f \in \mathbb{C}[y_0, \ldots, y_q]$ (*i.e.*, the F_j are algebraically dependent). We thus obtain

$$\mathbf{B}(\rho) = \{(F_0, \ldots, F_q) \in \prod_i \mathbb{P}\left(\mathbf{S}_{d_i}\right) \mid \underline{F} : \mathbb{C}^{r+1} \to \mathbb{C}^{q+1} \text{ is not dominant}\}. \tag{16}$$

For $q = 1$ the set theoretical description of ρ is rather simple:

$$\mathbf{B}(\rho) = \left\{(F_0, F_1) \in \mathbb{P}\left(\mathbf{S}_{d_0}\right) \times \mathbb{P}\left(\mathbf{S}_{d_1}\right) \mid F_0^{d_1} = F_1^{d_0}\right\}. \tag{17}$$

For general q we have a stratification

$$\mathbf{B}(\rho)_1 \subset \mathbf{B}(\rho)_2 \subset \cdots \subset \mathbf{B}(\rho)_q = \mathbf{B}(\rho)$$

where $\mathbf{B}(\rho)_k = \{(F_0, \ldots, F_q) \mid \dim \mathrm{image}(F) \le k\}$. The first stratum $\mathbf{B}(\rho)_1$ is set-theoretically equal to

$$\{(F_0, \ldots, F_q) \in \prod_i \mathbb{P}\left(\mathbf{S}_{d_i}\right) \mid F_0^{\hat{d}_0} = \ldots = F_q^{\hat{d}_q}\}$$

where $\hat{d}_j = \prod_{i \ne j} d_i$. For $k > 1$ the same set theoretical description is considerably more complex and we will carry it out only in very particular cases in §5.

Beware that the scheme structure of $\mathbf{B}(\rho)$ is often non-reduced, see §5.4.

At any rate, we register the following easy consequence of Lemma 2.1.

Proposition 4.1. *Let*

$$\begin{array}{rcl} \widetilde{\rho}: \prod_i \mathbb{P}(\mathbf{S}_{d_i}) & \dashrightarrow & \mathbb{P}\left(\mathbf{S}_{d-1} \otimes \bigwedge^{q+1} \mathbf{S}_1^\star\right) \\ (F_0, \dots, F_q) & \mapsto & dF_0 \wedge \dots \wedge F_q. \end{array}$$

Then the base loci of $\widetilde{\rho}$ and ρ are one and the same as schemes.

Proof. Let $V \subset \mathbf{S}_e \otimes \bigwedge^q \mathbf{S}_1^\star$ be the subspace of closed q–forms with coefficients of degree e. Put $W = i_R(V) \subset \mathbf{S}_{e+1} \otimes \bigwedge^{q-1} \mathbf{S}_1^\star$. Then $i_R : V \to W$ is a linear isomorphism. We still denote by $i_R : \mathbb{P}(V) \to \mathbb{P}(W)$ the projectivization. Since the image of $\widetilde{\rho}$ lies in $\mathbb{P}(V)$ and $\rho = i_R \circ \widetilde{\rho}$, the assertion follows. □

4.2. **Weighted homogeneous polynomials.** Fix $\bar{d} = (d_0, \dots, d_q) \in \mathbb{N}^{q+1}$ and $e \in \mathbb{N}$. A polynomial f in $\mathbb{C}[y_0, \dots, y_q]$ is said to be weighted homogeneous of type $\bar{d}$ and degree e if

$$f(\lambda^{d_0} y_0, \dots, \lambda^{d_q} y_q) = \lambda^e f(y_0, \dots, y_q)$$

for any $\lambda \in \mathbb{C}$. Equivalently, f is a linear combination of monomials

$$\prod_{0 \le j \le q} y_j^{\alpha_j} \quad \text{such that} \quad \bar{d} \cdot \alpha := \sum_{0 \le j \le q} d_j \alpha_j = e.$$

This is tantamount to declaring each variable y_i to be of degree d_i.

We denote by

$$\mathbf{S}_{q,\bar{d},e}$$

the $\mathbb{C}$-vector space of all such polynomials and write its dimension as $N(q, \bar{d}, e)$. Notice that $N(q, \bar{d}, e) = \dim \mathbf{S}_{q,\bar{d},e}$ can be expressed by the Hilbert series

$$H(t) = \sum_e N(q, \bar{d}, e) t^e = \frac{1}{\prod_{i=1}^q (1 - t^{d_i})}.$$

Throughout we will assume that the vector of natural numbers $\bar{d} \in \mathbb{N}^{q+1}$ is non-decreasingly ordered, *i.e.*, $d_0 \le d_1 \le \dots \le d_q$.

Define $\bar{e} = \bar{e}(\bar{d}) = (e_1, \dots, e_k)$ such that $e_i < e_{i+1}$ and $\cup_{0 \le i \le q} \{d_i\} = \cup_{1 \le i \le k} \{e_i\}$. If n_i stands for the number of times the natural number e_i appears in $\bar{d}$ then the pair $(\bar{e}, \bar{n})$, where $\bar{n} = (n_1, \dots, n_k)$, determines $\bar{d}$.

Set $q_j = -1 + \sum_{1 \le i \le j} n_i$, and for $l = 1, \dots, k$

$$\bar{d}_l = (\underbrace{e_1, \dots, e_1}_{n_1 \text{ times}}, \underbrace{e_2, \dots, e_2}_{n_2 \text{ times}}, \dots, \underbrace{e_l, \dots, e_l}_{n_l \text{ times}}).$$

Clearly, for each $f \in \mathbf{S}_{q,\bar{d},e_j}$, no variable y_i with weight $d_i > e_j$ occurs in f; thus

$$\mathbf{S}_{q,\bar{d},e_j} \cong \mathbf{S}_{q_j,\bar{d}_j,e_j}.$$

Denote by $\mathbb{E}^{q+1} = \mathrm{End}(\mathbb{C}^{q+1})$ the set of all polynomial maps $f : \mathbb{C}^{q+1} \to \mathbb{C}^{q+1}$. It is a ring under sum and composition of maps. If $f = (f_0, \dots, f_q) \in \mathbb{E}^{q+1}$, we say that f is of type $\bar{d}$ if f_i is weighted homogeneous of type $\bar{d}$ and degree d_i, for all $i = 0, \dots, q$.

Lemma 4.1. *Maps of type $\bar{d}$ form a subring of $\mathbb{E}^{q+1}$. More precisely, if $f, g \in \mathbb{E}^{q+1}$ are of type $\bar{d}$ then $f \circ g$ is of type $\bar{d}$. Moreover, the set*

$$\mathrm{GL}(q, \bar{d}) = \{f \in \mathbb{E}^{q+1} | f \text{ is of type } \bar{d} \text{ and } df(0) \text{ is invertible}\}$$

is a group.

Proof. $(f_i\circ g)(t^{d_0}y_0,\dots,t^{d_q}y_q) = f_i(g_0(t^{d_0}y_0,\dots,t^{d_q}y_q)),\dots,g_q(t^{d_0}y_0,\dots,t^{d_q}y_q)) = f_i(t^{d_0}g_0(y_0,\dots,y_q),\dots,t^{d_q}g_q(y_0,\dots,y_q)) = t^{d_i}f_i(g_0(y_0,\dots,y_q),\dots,g_q(y_0,\dots,y_q)) = t^{d_i}(f_i\circ g)(y_0,\dots,y_q)$,

We have $\mathrm{G} = \mathrm{GL}(q,\bar d)$ is closed under compositions. It remains to show that every element is invertible in G. Let us denote the block of variables of weight e_i by

$$\underline{y}_1 = \underbrace{y_0,\dots,y_{q_1}}_{\text{(weight } e_1)},\quad \underline{y}_2 = \underbrace{y_{q_1+1},\dots,y_{q_2}}_{\text{(weight } e_2)},\quad \dots,\quad \underline{y}_k = \underbrace{y_{q_{k-1}},\dots,y_{q_k}}_{\text{(weight } e_k)}.$$

The main point is that each $f\in\mathrm{G}$ has the following triangular shape,

$$(\underline{f}_1(\underline{y}_1),\underline{f}_2(\underline{y}_1,\underline{y}_2),\dots,\underline{f}_k(\underline{y}_1,\dots,\underline{y}_k)).$$

Here

$$\underline{f}_i(\underline{y}_1,\dots,\underline{y}_i) = (f_{i1}(\underline{y}_1,\dots,\underline{y}_2),\dots,f_{2n_i}(\underline{y}_1,\dots,\underline{y}_i)),$$

with

$$f_{ij}(\underline{y}_1,\dots,\underline{y}_i) = g_{ij}(\underline{y}_1,\dots,\underline{y}_{i-1}) + h_{ij}(\underline{y}_i)\in\mathbf{S}_{q_i,\bar d_i,e_i}$$

where $h_{ij}(\underline{y}_i)$ is in fact linear in the block of variables $\underline{y}_i$ of weight e_i. Indeed, since $e_{i+1}>e_i$, no $\underline{y}_{i+1}$ occurs in $\underline{f}_i$. Thus f can be written as

$$(\underline{h}_1(\underline{y}_1),\underline{h}_2(\underline{y}_2)+\underline{g}_2(\underline{y}_1),\dots,\underline{h}_k(\underline{y}_k)+\underline{g}_k(\underline{y}_1,\dots,\underline{y}_{k-1})).$$

Now we see that $df(0)$ is made up of blocks of the linear maps $\underline{h}_i = d\underline{h}_i:\mathbb{C}^{n_i}\to\mathbb{C}^{n_i}$. Hence invertibility of the former is equivalent to $d\underline{h}_i\in\mathrm{GL}_{n_i}\ \forall i$. Thus, given $(z_1,\dots,z_q)=(f(y))$, one can solve successively

$$\begin{cases} \underline{y}_1 = \underline{h}_1{}^{-1}(\underline{z}_1), \text{ then} \\ \underline{y}_2 = \underline{h}_2{}^{-1}(\underline{z}_2 - \underline{g}_2(\underline{y}_1)), \\ \vdots \\ \underline{y}_k = \underline{h}_k{}^{-1}(\underline{z}_k - \underline{g}_k(\underline{y}_1,\dots,\underline{y}_{k-1})). \end{cases}$$

□

The group $\mathrm{GL}(\mathrm{q},\bar{\mathrm{d}})$ naturally acts on the domain of μ (cf. 9):

$$\begin{array}{ccc} \mathrm{GL}(\mathrm{q},\bar{\mathrm{d}})\times\prod\limits_{0\le j\le q}\mathbf{S}_{d_j} & \longrightarrow & \prod\limits_{0\le j\le q}\mathbf{S}_{d_j} \\ (f,(F_0,\dots,F_q)) & \mapsto & (f_0(\underline{F}),\dots,f_q(\underline{F})). \end{array}$$

In other words, considering $\underline{F}$ as a polynomial map $\underline{F}:\mathbb{C}^{r+1}\to\mathbb{C}^{q+1}$, the action is just composition with a polynomial map $f:\mathbb{C}^{q+1}\to\mathbb{C}^{q+1}$ which belongs to $\mathrm{GL}(\mathrm{q},\bar{\mathrm{d}})$.

4.3. **The fibers of ρ.** The key tool for the description of the fiber of ρ and the proof of Theorem 3 is the following Proposition.

Proposition 4.2. *Let $\underline{F}=(F_0,\dots,F_q),\underline{G}=(G_0,\dots,G_q)\in\mathbf{S}_{d_0}\times\cdots\times\mathbf{S}_{d_q}$ Suppose that both $dF_0\wedge\cdots\wedge dF_q$ and $dG_0\wedge\cdots\wedge dG_q$ are non-zero $(q+1)$-forms. If* $\operatorname{codim}\operatorname{sing}(dF_0\wedge\cdots\wedge dF_q)\ge 2$ *then the following conditions are equivalent:*

(a) *$i_R(dF_0\wedge\cdots\wedge dF_q) = i_R(dG_0\wedge\cdots\wedge dG_q)$ up to a constant multiple.*
(b) *$dF_0\wedge\cdots\wedge dF_q = dG_0\wedge\cdots\wedge dG_q$ up to a constant multiple.*
(c) *$dG_j = \sum_{0\le k\le q}A_{jk}\,dF_k$ for some $A_{jk}\in\mathbf{S}_{d_j-d_k}$, for all j.*
(d) *$G_j = f_j(F_0,\dots,F_q)$ for some $f_j\in\mathbb{C}[y_0,\dots,y_q]$, for all j.*

(e) $G_j = f_j(F_0, \dots, F_q)$, *for all* j *for a unique* $f_j \in \mathbf{S}_{q,\bar{d},d_j}$. *Moreover,* $(f_0, \dots, f_q)$ *belongs to* $\mathrm{GL}(q, \bar{d})$.

Proof. (a) $\Leftrightarrow$ (b): Use the identity $d(i_R(dF_0 \wedge \cdots \wedge dF_q)) = (q+d)(dF_0 \wedge \cdots \wedge dF_q)$ from Lemma 2.1.

(b) $\Rightarrow$ (c): Multiplying by dG_j we obtain $dG_j \wedge dF_0 \wedge \cdots \wedge dF_q = 0$. Since $\underline{F}$ is generic, it follows by the division lemma that the dG_j are linear combinations of the dF_k. The coefficients may be chosen as homogeneous polynomials, necessarily of the stated degree.

(c) $\Rightarrow$ (b): Using the hypothesis and calculating wedges we have

$$dG_0 \wedge \cdots \wedge dG_q = \det(A)\ dF_0 \wedge \cdots \wedge dF_q.$$

Now $\det(A)$ is a non-zero homogeneous polynomial, and its degree is zero, so it is a constant, thereby proving the claim.

(d) $\Rightarrow$ (e): Let $f_j = \sum_\alpha c_\alpha y^\alpha$, where $\alpha \in \mathbb{N}^{q+1}$ and $c_\alpha \in \mathbb{C}$, so that $G_j = \sum_\alpha c_\alpha F^\alpha$. Write $f_j = g_j + h_j$ where g_j is the sum over the exponents α such that $\bar{d} \cdot \alpha = d_j$. We have $h_j(\underline{F}) = 0$ by the homogeneity of G_j and of the F_k. Therefore we may take $f_j = g_j$, the weighted homogeneous polynomial that we needed. Uniqueness is clear since the F_k are algebraically independent. Finally, setting $f = (f_0, \dots, f_q)$, since

$$dG_0 \wedge \cdots \wedge dG_q = \det(df) dF_0 \wedge \cdots \wedge dF_q$$

it follows that $\det(df) = \det(df(0))$ is a nonzero constant.

(e) $\Rightarrow$ (d): obvious.

(d) $\Rightarrow$ (c): If $G_j = \sum_\alpha c_\alpha F^\alpha$, taking exterior derivative we immediately get dG_j as a linear combination of the dF_k.

(c) $\Rightarrow$ (d): It suffices to use Lemma 4.2 below. $\square$

Lemma 4.2. *Let* $\underline{F} = (F_0, \dots, F_q) \in \mathbf{S}_{d_0} \times \cdots \times \mathbf{S}_{d_q}$ *be generic. Let* G *be a homogeneous polynomial of degree* e *such that* $dG = \sum_{0 \le k \le q} A_k\, dF_k$ *for some* $A_k \in \mathbf{S}_{e-d_k}$. *Then* $G = f(F_0, \dots, F_q)$ *for a unique polynomial* $f \in \mathbf{S}_{q,\bar{d},e}$.

Proof. We proceed by induction on e. The assertion is clear for $e = 0$. Taking exterior derivative we have $d^2G = \sum_k dA_k \wedge dF_k = 0$. Thus $dA_k \wedge dF_0 \wedge \cdots \wedge dF_q = 0$ for all k. Since $\underline{F}$ is generic, we get $dA_k = \sum_h B_{kh}\, dF_h$ for some $B_{kh} \in \mathbf{S}_{e-d_k-d_h}$. By the inductive hypothesis, $A_k = f_k(F_0, \dots, F_q)$ for some polynomial f_k. On the other hand, applying i_R to $dG = \sum_k A_k\, dF_k$ we obtain $eG = \sum_k A_k\, d_k F_k$. Replacing here A_k by $f_k(F_0, \dots, F_q)$ we obtain the claim. Uniqueness and weighted homogeneity were argued before. $\square$

Proposition 4.3. *For general* $\underline{F} = (F_0, \dots, F_q) \in \prod_{0 \le j \le q} \mathbf{S}_{d_j}$ *we have a bijective map*

$$\begin{array}{ccc} \mathrm{GL}(q, \bar{d}) & \longrightarrow & \mu^{-1}\mu(\underline{F}) \\ (f_0, \dots, f_q) & \longmapsto & (f_0(\underline{F}), \dots, f_q(\underline{F})) \end{array}$$

with μ *the multilinear map inducing* ρ *as in* (9).

Proof. The assertion follows from the equivalence (a) $\Longleftrightarrow$ (e) in 4.2. $\square$

Corollary 4.1. *We have the formula for the fiber dimension,*

$$\dim \rho^{-1}\rho(\underline{F}) = \sum_{0 \le j \le q} (N(q, \bar{d}, d_j) - 1).$$

4.4. **A natural factorization and proof of Theorem 3.** We will now proceed to describe a tower of open subsets of Grassmann bundles birational to $\mathcal{R}(r,\bar{d})$. We preserve the notation of Subsection 4.2.

Start with $Y_0 = G(n_1, \mathbf{S}_{e_1})$, the grassmannian of n_1-planes in $\mathbf{S}_{e_1}$. Let $X_1 \subset Y_1$ be the open subset defined as

$$X_1 = \{F_1 \wedge \cdots \wedge F_{n_1} \in G(n_1, S_{e_1}) \,|\, \operatorname{codim} \operatorname{sing}(dF_0 \wedge \cdots \wedge dF_{n_1}) \geq 2\}.$$

Now let $\mathcal{A}_2 \to X_1$ be the vector subbundle of the trivial bundle $\mathbf{S}_{e_2} \times X_1$ with fiber over $\underline{F}_1 = F_1 \wedge \cdots \wedge F_{n_1} \in X_1$ given by

$$\mathcal{A}_2(\underline{F}_1) = \{G \in \mathbf{S}_{e_2} \,|\, dF_1 \wedge \cdots \wedge dF_{n_1} \wedge dG = 0\}.$$

Recalling Lemma 2.2(a), and the above considerations on weighted homogeneity, we have in fact

$$\mathcal{A}_2(\underline{F}_1) = \{G \in \mathbf{S}_{e_2} \,|\, G = f(\underline{F}_1), f \in \mathbf{S}_{q_1,\bar{d}_1,e_2}\} \cong \mathbf{S}_{q_1,\bar{d}_1,e_2}.$$

Let $Y_2 = G(n_2, \mathbf{S}_{e_2}/\mathcal{A}_2)$ be the Grassmann bundle over X_1. Notice that, for an element $\underline{G}_2 = [G_1] \wedge \cdots \wedge [G_{n_2}] \in G(n_2, \mathbf{S}_{e_2}/\mathbf{S}_{q,\bar{d},e_2}(p))$ over a point $\underline{F}_1 = F_1 \wedge \cdots \wedge F_{n_1} \in X_1$, the (n_1+n_2)-form

$$\eta(\underline{G}_2) = dF_1 \wedge \cdots dF_{n_1} \wedge dG_1 \wedge \cdots \wedge dG_{n_2}$$

is well-defined up to a non zero multiplicative constant. Therefore we can set $X_2 \subset Y_2$ as the open subset defined by

$$X_2 = \{\underline{G}_2 \in Y_2 \,|\, \operatorname{codim} \operatorname{sing} \eta(\underline{G}_2) \geq 2\}$$

Continuing, we have a vector subbundle $\mathcal{A}_3$ of $\mathbf{S}_{e_3} \times X_2$ with fiber

$$\mathcal{A}_3(\underline{F}_1, \underline{G}_2) = \{H \in \mathbf{S}_{e_3} \,|\, dF_1 \wedge \cdots \wedge dF_{n_1} \wedge dG_1 \wedge \cdots \wedge dG_{n_2} \wedge dH = 0\}.$$

As before, this is isomorphic to $\mathbf{S}_{q_2,\bar{d}_2,e_3}$. Proceeding this way, we arrive at an open subset $X = X_k \subset Y_k$ where $Y_k \to X_{k-1}$ is the Grassmann bundle $G(n_k, \mathbf{S}_{e_k}/\mathcal{A}_{k-1})$. Clearly X is a rational variety just like all Grassmann bundles over rational varieties. Using Proposition 4.2, we arrive at a birrational map from X to $\mathcal{R}(r,\bar{d})$. It follows that $\mathcal{R}(r,\bar{d})$ is rational and this concludes the proof of Theorem 3 □

5. Degree calculations

Let $\bar{d} = (d_0, \ldots, d_q)$, $\bar{e}$, $\bar{n}$, ... be as in the previous section. Here we proceed to find the degree of the projective variety

$$\mathcal{R}(r,\bar{d}) \subset \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^q(d+q+1))\right)$$

in some cases. We shall time and again profit from the following consequence of Proposition 4.1. We consider

$$\widetilde{\rho} : \prod_i \mathbb{P}\left(\mathbf{S}_{d_i}\right) \dashrightarrow \widetilde{\mathcal{R}}(r,\bar{d}) = (i_R)^{-1}\mathcal{R}(r,\bar{d}) \subset \mathbb{P}\left(\mathbf{S}_{d-1} \otimes \overset{q+1}{\wedge} \mathbf{S}_1^\star\right).$$

Thus we see that all degree calculations can be lifted from $\mathbb{P}(W) \subset \mathbb{P}\left(\mathrm{H}^0(\mathbb{P}^r, \Omega^q(d+q+1))\right)$ to $\mathbb{P}(V)$.

5.1. **Linear projections of grassmannians.** When $q_1 = q$, i.e. all the degrees d_i are equal to e_1, the variety X constructed in §4.4 is an open subset of the grassmannian $G(q, \mathbf{S}_{e_1})$. It follows that the morphism $\bar{\rho} : X \to \mathcal{R}(r, \bar{d})$ gives rise to a rational map

$$\widetilde{\rho} : G(q+1, \mathbf{S}_{e_1}) \dashrightarrow \widetilde{\mathcal{R}}(r, \bar{d}) \subset \mathbb{P}\left(\mathbf{S}_{d-1} \otimes \overset{q+1}{\wedge} \mathbf{S}_1^\star\right).$$

Notice that $\bar{\rho}$ is the composition of Plücker's embedding with a central projection

$$\begin{array}{ccc} \mathbb{P}\left(\bigwedge^{q+1} S_{e_1}\right) & \dashrightarrow & \mathbb{P}\left(\mathbf{S}_{d-1} \otimes \overset{q+1}{\wedge} \mathbf{S}_1^\star\right) \\ F_0 \wedge \cdots \wedge F_q & \mapsto & dF_0 \wedge \cdots \wedge dF_q. \end{array}$$

It is a simple exercise to show that $G(q+1, \mathbf{S}_{e_1})$ is disjoint from the center of this projection if, and only if, $q = 1$ or $d_0 = \cdots = d_q = 1$. In both cases the degree of these components is equal to the degree of the corresponding grassmannians under Plücker's embedding (see e. g. [12]). More precisely, setting $N = (q+1)(r-q) = \dim G(q+1, r+1)$, we have

$$\boxed{\begin{array}{lclcl} \deg(\mathcal{R}(q,1,\ldots,1)) & = & \deg G(q+1, \mathbf{S}_1) & = & \frac{1!2!\cdots q!N!}{(r-q)!(r-q+1)!\ldots r!} \\ \deg(\mathcal{R}(1,d,d)) & = & \deg G(2, \mathbf{S}_{d_1}) & = & \frac{1}{N_d-1}\binom{2N_d-2}{N_d}, \\ \text{where} & & N_d = \binom{r+d}{r} - 1. & & \end{array}} \tag{18}$$

Remark 5.1. The scheme-theoretic structure of the base locus of a rational map $\phi : Y \dashrightarrow \mathbb{P}\left(\mathbb{C}^N\right)$ is defined as follows (cf. [9, 7.17.3, p. 168]). We are given a line bundle (=invertible sheaf) $\mathcal{L}$ over Y together with a homomorphism $\mathcal{O}_Y^N \to \mathcal{L}$, surjective over the open dense subset $U \subseteq Y$ where ϕ is a morphism. The image, $\mathcal{J}$, of the induced homomorphism

$$\begin{tikzcd} \mathcal{O}_Y^N \otimes \mathcal{L}^\vee \arrow[r] \arrow[dr] & \mathcal{O}_Y \\ & \mathcal{J} \arrow[u, hook] \end{tikzcd}$$

is the sheaf of ideals defining the base locus. If D denotes an effective Cartier divisor such that $\mathcal{J} = \mathcal{O}_Y(-D) \cdot \mathcal{J}'$ for some ideal sheaf $\mathcal{J}'$, then the set of zeros, $V(\mathcal{J}')$ is contained in $V(\mathcal{J})$. Clearly ϕ extends to the complement $U' = Y \setminus V(\mathcal{J}') \supseteq U$ in such a way that the pullback of the hyperplane bundle is

$$\phi^\star_{|U} \mathcal{O}_{\mathbb{P}(\mathbb{C}^N)}(1) = \mathcal{L} \otimes \mathcal{O}(-D).$$

5.2. **(2,2,2).** When $q = 2$ and $d_0 = d_1 = d_2 = 2$ the situation is still manageable. It turns out that the indeterminacy locus of the rational map

$$\begin{array}{ccc} \widetilde{\rho} : X = G(3, \mathbf{S}_2) & \dashrightarrow & \widetilde{\mathcal{R}}(r, \bar{d}) \subset \mathbb{P}(\mathbf{S}_3 \otimes \overset{3}{\wedge} \mathbf{S}_1^\star) \\ F_0 \wedge F_1 \wedge F_2 & \mapsto & dF_0 \wedge dF_1 \wedge dF_2 \end{array}$$

is schematically equal to the image of the Veronese-like embedding

$$\begin{array}{ccc} Y = G(2, \mathbf{S}_1) & \xhookrightarrow{\text{v}} & X = G(3, \mathbf{S}_2) \\ \langle L_0, L_1 \rangle & \longmapsto & \langle L_0^2, L_0 L_1, L_1^2 \rangle. \end{array}$$

Thus a single blowup $\pi : \widetilde{X} \to X$ along Y resolves the indeterminacy *i.e.*, the induced map $\widetilde{\rho} : \widetilde{X} \to \widetilde{\mathcal{R}}(r, \bar{d})$ is a morphism. Indeed, write the tautological sequence of $G(3, \mathbf{S}_2)$

$$R_2 \rightarrowtail \mathbf{S}_2 \twoheadrightarrow Q_2 \tag{19}$$

and likewise for $G(2, \mathbf{S}_1)$,

$$R_1 \rightarrowtail \mathbf{S}_1 \twoheadrightarrow Q_1. \tag{20}$$

The fiber of R_2 over $\underline{F} \in X$ is the space $\langle F_0, F_1, F_2\rangle$ spanned by three independent quadratic forms. In order to find the pullback of the hyperplane class via the resolved map

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[rr, bend left, "\widetilde{\rho}\pi"] & X \arrow[r, dashed] & \widetilde{\mathcal{R}}(r, \bar{d}), \end{tikzcd}$$

we have at first

$$\begin{tikzcd} \widetilde{\rho}^{\star}\mathcal{O}(-1) = \overset{3}{\wedge} R_2 \arrow[r] \arrow[dr] & \overset{3}{\wedge} \mathbf{S}_2 \arrow[d] & \ni & F_0 \wedge F_1 \wedge F_2 \arrow[d, mapsto] \\ & \mathbf{S}_3 \otimes \overset{3}{\wedge} \mathbf{S}_1^{\star} & \ni & dF_0 \wedge dF_1 \wedge dF_2. \end{tikzcd}$$

The indeterminacy locus, $Z \subset X$, of $\widetilde{\rho} : X \dashrightarrow \widetilde{\mathcal{R}}(r, \bar{d})$ is the scheme of zeros of the slant arrow, $\overset{3}{\wedge} R_2 \longrightarrow \mathbf{S}_3 \otimes \overset{3}{\wedge} \mathbf{S}_1^{\star}$. Dualizing, we find $\overset{3}{\wedge} R_2^{\star} \longleftarrow (\mathbf{S}_3 \otimes \overset{3}{\wedge} \mathbf{S}_1^{\star})^{\star}$, whence the ideal sheaf of Z appears as the image

$$(\mathbf{S}_3 \otimes \overset{3}{\wedge} \mathbf{S}_1^{\star})^{\star} \otimes \overset{3}{\wedge} R_2 \twoheadrightarrow I(Z) \subset \mathcal{O}_X. \tag{21}$$

We claim that Z is equal to the image of $\mathrm{v} : G(2, \mathbf{S}_1) \hookrightarrow G(3, \mathbf{S}_2)$. Indeed, first note that Z is invariant under linear change of coordinates in $\mathbb{P}^r$. Since it is closed, it must contain a closed orbit of $G(3, \mathbf{S}_2)$. There are just two closed orbits, to wit those given by the representatives: $\langle x_0^2, x_0x_1, x_0x_2\rangle$ and $\langle x_0^2, x_0x_1, x_1^2\rangle$. Only the latter one lies in Z. The calculation of the tangent space to Z at the point $\langle x_0^2, x_0x_1, x_1^2\rangle$ performed below shows that Z is of dimension at most $2(r-1)$. Since Z contains the image of $G(2, \mathbf{S}_1)$, it is in fact smooth and equal to that image. The tangent space is given by the equation

$$\begin{gathered} d(x_0^2 + \varepsilon F_0) \wedge d(x_0x_1 + \varepsilon F_1) \wedge d(x_1^2 + \varepsilon F_2) = \\ 2\varepsilon dx_0 \wedge dx_1 \wedge (x_0^2 dF_2 - 2x_0x_1 dF_1 + x_1^2 dF_0) = 0, \end{gathered}$$

where the $F_i \in \mathbf{S}_2 / \langle x_0^2, x_0x_1, x_1^2\rangle$.

Equivalently:

$$\begin{cases} x_0^2 \dfrac{\partial F_2}{\partial x_2} - 2x_0x_1 \dfrac{\partial F_1}{\partial x_2} + x_1^2 \dfrac{\partial F_0}{\partial x_2} = \dfrac{\partial}{\partial x_2}(x_0^2 F_2 - 2x_0x_1F_1 + x_1^2 F_0) = 0, \\ \qquad\qquad\qquad\vdots \\ x_0^2 \dfrac{\partial F_2}{\partial x_r} - 2x_0x_1 \dfrac{\partial F_1}{\partial x_r} + x_1^2 \dfrac{\partial F_0}{\partial x_r} = \dfrac{\partial}{\partial x_r}(x_0^2 F_2 - 2x_0x_1F_1 + x_1^2 F_0) = 0. \end{cases}$$

We'd like to deduce that the subspace consisting of triples

$$(F_0, F_1, F_2) \in \left(\mathbf{S}_2 / \langle x_0^2, x_0x_1, x_1^2\rangle\right)^{\oplus 3}$$

defined by the system just above must be of dimension

$$\dim G(2,\mathbf{S}_1)=2(r-1).$$

We see that $x_0^2F_2-2x_0x_1F_1+x_1^2F_0$ is independent of $x_2,\dots,x_r$. Thus, no monomial x_mx_n, $2\le m,n\le r$ appears in the F_i. It follows that the F_i are of the form

$$F_i=a_{i0}x_0+a_{i1}x_1$$

with the $a_{ij}\in\mathbb{C}[x_2,\dots,x_r]$ homogeneous of degree one. We have then

$$\begin{gathered}x_0^2(a_{20}x_0+a_{21}x_1)-2x_0x_1(a_{10}x_0+a_{11}x_1)+x_1^2(a_{00}x_0+a_{01}x_1)=\\ a_{20}x_0^3+(a_{21}-2a_{10})x_0^2x_1+(a_{00}-2a_{11})x_0x_1^2+a_{01}x_1^3\in\mathbb{C}[x_0,x_1].\end{gathered}$$

This implies

$$a_{20}=a_{21}-2a_{10}=a_{00}-2a_{11}=a_{01}=0.$$

Hence the F_i depend exactly on $2(r-1)$ parameters. This achieves the verification that $Z=\mathrm{v}(G(2,\mathbf{S}_1))$.

Pulling back the surjection (21) to the blowup $\pi:\widetilde{X}\to X$, we find the surjections

$$\pi^\star(\mathbf{S}_3\otimes\overset{3}{\wedge}\mathbf{S}_1^\star)^\star\otimes\overset{3}{\wedge}R_2\twoheadrightarrow\pi^\star I(Z)\twoheadrightarrow\mathcal{O}_{\widetilde{X}}(-E)=I(E),$$

with $E=\pi^{-1}Z$, the exceptional divisor. This yields the formula

$$\widetilde{\rho}\mathcal{O}_{\widetilde{\mathbb{F}}}(1)=\pi^\star\overset{3}{\wedge}R_2^\star\otimes\mathcal{O}_{\widetilde{X}}(-E).$$

It follows that the pullback of the hyperplane class is given by

$$\widetilde{\rho}^\star\mathbf{h}=\pi^\star\mathbf{q}_1-E,$$

where $\mathbf{q}_1=c_1Q_2$ (see 19). Since $\widetilde{\rho}$ is generically injective, the degree of the image can be calculated as

$$\deg\mathcal{R}(r,2,2,2)=\int_{\widetilde{X}}\widetilde{\rho}^\star\mathbf{h}^{\dim X}.$$

Setting $N=\dim X=\dim G(3,\mathbf{S}_2)=3(\binom{r+2}{2}-3)$, we see that the degree is given by

$$\int_{\widetilde{X}}\widetilde{\rho}^\star\mathbf{h}^N=\int_X\pi_\star\sum_0^N\binom{N}{i}\pi^\star\mathbf{q}_1^i\cdot(-E)^{N-i}.$$

Using projection formula, we are reduced to the calculation of

- the Plücker's degree of $G(3,\mathbf{S}_2)$ for the term with $i=N$,

and

- the contribution of $\pi_\star(E)^j=(-1)^{j-1}\mathrm{v}_\star s_{j-\delta}\mathcal{N}$,

where $\mathcal{N}$ stands for the normal bundle of the embedding v and

$$\delta=\operatorname{rank}\mathcal{N}=\dim G(3,\mathbf{S}_2)-\dim G(2,\mathbf{S}_1).$$

The minus signs come from the formula

$$\iota^\star\mathcal{O}_{\widetilde{X}}(E)=\mathcal{O}_{\mathcal{N}}(-1).$$

The Segre classes of the normal bundle are obtained from the usual exact sequence

$$\begin{array}{ccccc} TY & \rightarrowtail & TX_{|Y} & \twoheadrightarrow & \mathcal{N}\\ \| & & \| & & \\ \mathrm{Hom}(R_2,Q_2) & & \mathrm{v}^\star\mathrm{Hom}(R_3,Q_3) & & \end{array}\tag{22}$$

By definition of v, we have $\mathrm{v}^\star R_3=\mathrm{Sym}_2R_2$. Using SCHUBERT [11], we find,

r	deg
3	1324220
4	2860923458080
5	243661972980477736263
6	728440733705107831789517245858
7	704613096513585123585398408696231899176183

$\boxed{d_0 = d_1 = d_2 = 2}$

A maple script is available at [16].

5.3. **Bundles of projective spaces.** When $k = 2$ and $n_2 = 1$, the variety X constructed in §4.4 is an open subset of a projective bundle over an open subset of a grassmannian. In general we do not know a manageable compactification. Even when we can compactify X as above, the scheme structure of the base locus of $\bar{\rho}$ can be non reduced and is far form being understood in general.

Nevertheless in the following three cases we are able to handle the degree:

- $q = 1$ and d_0 divides d_1.
- arbitrary q but $k = 2$ and $d_1 = 1$, *i.e.*, $\bar{d} = (1, \ldots, 1, e)$.
- $q = 1$, $d_0 = 2$ and $d_1 = 3$.

5.3.1. *First Case: $q = 1$ and d_0 divides d_1.* This is in fact the only case for which we got a closed formula. Now the natural parameter space is the projective bundle

$$X \longrightarrow \mathbb{P}(\mathbf{S}_{d_0})$$

described in the sequel.

Write the tautologic line subbbundle over $\mathbb{P}(\mathbf{S}_{d_0})$,

$$\mathcal{O}_{\mathbf{S}_{d_0}}(-1) \rightarrowtail \mathbf{S}_{d_0}.$$

Set $\kappa = d_1/d_0$. Taking symmetric power, we have the exact sequence

$$\mathcal{O}_{\mathbf{S}_{d_0}}(-\kappa) \rightarrowtail \mathbf{S}_{d_1} \twoheadrightarrow \overline{\mathbf{S}}_{d_1},$$

which defines the vector bundle $\overline{\mathbf{S}}_{d_1}$. The fiber of $\overline{\mathbf{S}}_{d_1}$ over each $F_0 \in \mathbb{P}(\mathbf{S}_{d_0})$ is the quotient vector space $\mathbf{S}_{d_1}/\langle F_0^{\kappa}\rangle$. Thus we have

$$\begin{array}{rcl} \widetilde{\rho} : X = \mathbb{P}\left(\overline{\mathbf{S}}_{d_1}\right) & \longrightarrow & \widetilde{\mathcal{R}}(r, d_0, d_1) \subseteq \mathbb{P}\left(\mathbf{S}_{d_1+d_0-2} \otimes \overset{2}{\wedge} \mathbf{S}_1^{\star}\right). \\ (F_0, \overline{F}_1) & \longmapsto & dF_0 \wedge dF_1. \end{array}$$

The pullback of the hyperplane class via the map $\widetilde{\rho}$ is obtained as follows. Form the diagram

$$\begin{tikzcd} \mathcal{O}_{\mathbf{S}_{d_0}}(-1) \otimes \mathbf{S}_{d_1} \arrow[r, tail] \arrow[dr, "\alpha"'] & \mathbf{S}_{d_0} \otimes \mathbf{S}_{d_1} \arrow[d] \\ & \mathbf{S}_{d_1+d_0-2} \otimes \overset{2}{\wedge} \mathbf{S}_1^{\star} \end{tikzcd} \tag{23}$$

where the vertical map is defined by

$$F_0 \otimes F_1 \mapsto dF_0 \wedge dF_1.$$

Composing the slant arrow α with the natural homomorphism

$$\mathcal{O}_{\mathbf{S}_{d_0}}(-1) \otimes \mathcal{O}_{\mathbf{S}_{d_0}}(-\kappa) \rightarrowtail \mathcal{O}_{\mathbf{S}_{d_0}}(-1) \otimes \mathbf{S}_{d_1}$$

we get zero since $dF_0 \wedge d(F_0^\kappa) = 0$. Hence α passes to the quotient,

$$\begin{tikzcd} \mathcal{O}_{\mathbf{S}_{d_0}}(-1)\otimes \mathbf{S}_{d_1} \arrow[r, two heads] \arrow[rr, bend left, "\alpha"] & \mathcal{O}_{\mathbf{S}_{d_0}}(-1)\otimes \overline{\mathbf{S}}_{d_1} \arrow[r, "\bar\alpha"'] & \mathbf{S}_{d_1+d_0-2}\otimes \overset{2}{\wedge}\mathbf{S}_1^\star. \end{tikzcd}$$

Composing $\bar\alpha$ with

$$\mathcal{O}_{\mathbf{S}_{d_0}}(-1)\otimes \mathcal{O}_{\overline{\mathbf{S}}_{d_1}}(-1) \rightarrowtail \mathcal{O}_{\mathbf{S}_{d_0}}(-1)\otimes \overline{\mathbf{S}}_{d_1}$$

we finally find the line subbundle,

$$\mathcal{O}_{\mathbf{S}_{d_0}}(-1)\otimes \mathcal{O}_{\overline{\mathbf{S}}_{d_1}}(-1) \rightarrowtail \mathbf{S}_{d_1+d_0-2}\otimes \overset{2}{\wedge}\mathbf{S}_1^\star.$$

The last map is injective at the point $(x_1^{d_0}, \overline{x_1^{d_1-1}x_2})$, which is a representative of the unique closed orbit of $\mathbb{P}\left(\overline{\mathbf{S}}_{d_1}\right)$. Hence it is injective everywhere. Alternatively, since F_0^κ, F_1 are linearly independent, the rational map $\mathbb{P}^r \dashrightarrow \mathbb{P}^1$ they define is non-constant, hence $dF_0 \wedge dF_1 \neq 0$. Thus, the pullback to X of the hyperplane class of the projective space $\mathbb{P}\big(\mathbf{S}_{d_1+d_0-2}\otimes \overset{2}{\wedge}\mathbf{S}_1^\star\big)$ is

$$H = \mathbf{h} + \mathbf{h}'$$

where $\mathbf{h} = c_1\mathcal{O}_{\mathbf{S}_{d_0}}(1)$, which comes from the base $\mathbb{P}(\mathbf{S}_{d_0})$, and $\mathbf{h}' = c_1\mathcal{O}_{\overline{\mathbf{S}}_{d_1}}(1)$, the relative hyperplane class. With the notation as in (18), we have

$$\operatorname{rank}\overline{\mathbf{S}}_{d_1} - 1 = N_{d_1} - 2$$

for the fiber dimension of $\mathbb{P}\left(\overline{\mathbf{S}}_{d_1}\right) \to \mathbb{P}(\mathbf{S}_{d_0})$. The sought for degree is

(24)
$$\begin{aligned} \deg \mathcal{R}(r,d_0,d_1) = & \int_{\mathbb{P}(\overline{\mathbf{S}}_{d_1})} H^{N_{d_1}+N_{d_0}-1} = \sum_i \binom{N_{d_1}+N_{d_0}-1}{i} \mathbf{h}^i s_{N_{d_0}-i}(\overline{\mathbf{S}}_{d_1}) \\ = & \binom{N_{d_1}+N_{d_0}-1}{N_{d_0}} - \frac{d_1}{d_0}\binom{N_{d_1}+N_{d_0}-1}{N_{d_0}-1}. \end{aligned}$$

The last equality follows from the calculation of the Segre class $s(\overline{\mathbf{S}}_{d_1}) = 1 - \kappa\mathbf{h}$, so $s_i(\overline{\mathbf{S}}_{d_1})$ is zero in degrees $i \geq 2$.

If $r = 3$, $d_1 = 2$, $d_0 = 1$, one finds $\binom{3+8}{3} - 2\binom{11}{2} = 55$. By constrast, the degree of the Segre variety $\check{\mathbb{P}}^3 \times \mathbb{P}^9 \subset \mathbb{P}^{39}$ of which the image of ρ is a rational projection, is equal to $\binom{12}{3}$.

5.3.2. *Second case: $k = 2$ and $d_0 = 1$.* We are now looking at foliations defined by $\omega = i_R(dF_0 \wedge \cdots \wedge dF_q)$ where $\deg F_0 = \cdots = \deg F_{q-1} = 1$; $\deg F_q = d \geq 2$. A natural parameter space is the projective bundle over the grassmannian $G = G(q, \mathbf{S}_1)$ defined as follows. Write the tautological sequence

$$R_q \rightarrowtail \mathbf{S}_1 \twoheadrightarrow Q.$$

The fiber of R_q over $\underline{F} \in G$ is the space $\langle F_0, \ldots, F_{q-1}\rangle$ spanned by linear forms. Now the last polynomial F_q is taken as a class in the projective space $\mathbb{P}\left(\mathbf{S}_d/\langle F_0^d, F_0\cdot F_1^{d-1}, \ldots, F_{q-1}^d\rangle\right)$. The natural homomorphism $\operatorname{Sym}_d R_q \to \mathbf{S}_d$ is injective; it corresponds to an instance of the vector bundle $\mathcal{A}_2$ described in 4.4. Form the projective bundle

$$\pi : X = \mathbb{P}(\mathbf{S}_d/\operatorname{Sym}_d R_q) \longrightarrow G.$$

Note that the rational map

$$\begin{array}{ccc} X & \overset{\bar{\rho}}{\dashrightarrow} & \mathbb{P}(\mathbf{S}_{d-1} \otimes \overset{q}{\wedge} \mathbf{S}_1^{\star}) \\ (\langle F_0, \ldots, F_{q-1} \rangle, \overline{F}_q) & \longmapsto & dF_0 \wedge \cdots \wedge dF_{q-1} \wedge dF_q \end{array}$$

is in fact regular everywhere. Indeed, regularity is an open condition; the map is invariant under the natural action of GL_{r+1} and is regular at the representative $(\langle x_0, \ldots, x_{q-1} \rangle, \overline{x_q^{d-1} x_0})$ of the unique closed orbit. Thus the sought for degree can be computed by Schubert calculus in the following manner. Set

$$\begin{array}{l} g = q(r+1-q) = \dim G \\ N = \binom{r+d}{r} - \binom{q-1+d}{q-1} - 1, \end{array} \tag{25}$$

so that presently the dimension of the component is $\delta = N + g$. The pullback of the hyperplane class from $\mathbb{P}(\mathbf{S}_{d-1} \otimes \overset{q}{\wedge} \mathbf{S}_1^{\star})$ is equal to $\mathbf{h} + \mathbf{q}_1$, where $\mathbf{h}$ stands for the relative hyperplane class of the projective bundle $X \to G$ and $\mathbf{q}_1 = c_1 Q$. By general principles, the degree is given by

$$\int_X (\mathbf{h} + \mathbf{q}_1)^{\delta} = \sum_0^g \binom{\delta}{i} \int_G \pi_{\star}(\mathbf{h}^{\delta - i}) \mathbf{q}_1^i = \sum_0^g \binom{\delta}{i} \int_G s_{g-i} \cdot \mathbf{q}_1^i.$$

Here $s_i = c_i(\mathrm{Sym}_d R)$. For $q = 2$, $r = 3$ we find

$$d^2(d-1)(d+3)(d^2+2)(d^2+4d+6)(d+2)^2(d+1)^2 \big/ (2^6 \cdot 3^5),$$

a polynomial of degree 12 in d. For $q = 2$; $r = 4, 5, 6, 7, 8$ we find polynomial formulas of respective degrees $24, 40, 60, 84, 112$. This suggests a polynomial degree like $2r(r-1)$. Now for $q = 3$, $r = 4, 5, 6, 7, 8$ we get polynomial formulae of degrees $3r(r-2)$ with respect to d. Further experiments (cf. [16]) suggest polynomial formulas of degrees $qr(r-q+1)$. Here is a sample for small values of r, q, d.

$(r, q) = (5, 2)$

d	2	3	4	5
deg	2390850	10457430102	9654013512864	3099059696318355

$(r, q) = (6, 2)$

d	2	3	4	5
deg	1139133688	91451421683006	1118409272891730904	3524857658574891999976

$(r, q) = (6, 3)$

2	3	4	5
8983484048	9350781792221835	1060759743612735149417	22044166363067583367287424

5.4. $(2, 2m+1)$**.** Assume $q = 1$, $d_0 = 2$ and $d_1 = 3$. Set for short $\mathbf{X} = \mathbb{P}(\mathbf{S}_2) \times \mathbb{P}(\mathbf{S}_3)$. Put as before $N_d = \binom{r+d}{d} - 1$. We have

$$\dim \mathbf{X} = N_2 + N_3.$$

We look closer at the indeterminacy locus of

$$\begin{array}{rcc} \widetilde{\rho} : \mathbf{X} & \dashrightarrow & \mathbb{P}\left(\mathbf{S}_3 \otimes \overset{2}{\wedge} \mathbf{S}_1^{\star} \right) \\ (F, G) & \mapsto & dF \wedge dG. \end{array}$$

It is, set-theoretically,

$$\mathbf{B}(\widetilde{\rho}) = \{ (L^2, L^3) \mid L \in \mathbb{P}(\mathbf{S}_1) \}.$$

Lemma 5.1. *The tangent space to the scheme of indeterminacy* $\mathbf{B} = \mathbf{B}(\widetilde{\rho})$ *is the subspace*

$$\{(F', G') \in T_{(L^2, L^3)}\mathbf{X} = \mathbf{S}_2/\langle L^2\rangle \oplus \mathbf{S}_3/\langle L^3\rangle \,|\, G' = \frac{3}{2}F'\}.$$

Proof. The tangent space to the scheme of indeterminacy is the set of pairs (F', G') such that $d(L^2 + \varepsilon F') \wedge d(L^3 + \varepsilon G') = 0$. Expanding we get

$$2dL \wedge dG' + 3LdF' \wedge dL = dL \wedge (2dG' - 3LdF') = 0. \tag{26}$$

By division, we must have $2dG' - 3LdF' = F''dL$ for some $F'' \in \mathbf{S}_2$. This implies $dF'' \wedge dL = 3dF' \wedge dL$. Hence again by division, $dF'' - 3dF' = AdL$ for some $A \in \mathbf{S}_1$. This implies $A = aL$ for some constant c. Thus $d(F'' - 3F' - \frac{1}{2}aL^2) = 0$ so that in fact $F'' = 3F' + \frac{1}{2}aL^2$. Plugging back in a previous relation, we find $2dG' - 3LdF' = (3F' + \frac{1}{2}aL^2)dL$ whence $2dG' - \frac{1}{6}adL^3 = 3d(LF')$. This yields $2G' - \frac{1}{6}aL^3 = 3LF'$, hence $G' = \frac{3}{2}LF'$ in $\mathbf{S}_3/\langle L^3\rangle$. Conversely, it is easy to see that for such $G' = \frac{3}{2}LF'$, the differential form $2dG' - 3LdF'$ is a multiple of dL, hence (26) holds. ☐

Set $\mathbf{V} = \mathbf{B}_{red} \cong \mathbb{P}(\mathbf{S}_1)$. Thus $\mathbf{B}$ is a multiple structure or thickenning of $\mathbf{V}$. The tangent sheaf to $\mathbf{B}$ is in fact a vector bundle of rank $\dim \mathbb{P}(\mathbf{S}_2)$. We have the exact sequence of vector bundles over $\mathbf{V}$,

$$T\mathbf{V} \rightarrowtail T\mathbf{B}_{|\mathbf{V}} \twoheadrightarrow \mathcal{N}_{\mathbf{V}/\mathbf{B}}$$

where $\mathcal{N}_{\mathbf{V}/\mathbf{B}}$ stands for the normal bundle of $\mathbf{V} \subset \mathbf{B}$. We register the formula

$$\operatorname{rank} \mathcal{N}_{\mathbf{V}/\mathbf{B}} = \binom{r+2}{2} - r = \binom{r+1}{2} + 1.$$

We look at the blowup $\mathbf{X}' \to \mathbf{X}$ along $\mathbf{V}$. Denote by $\mathbf{E} \subset \mathbf{X}'$ the exceptional divisor. Recall we have $\mathbf{E}' = \mathbb{P}\left(\mathcal{N}_{\mathbf{V}/\mathbf{X}}\right)$, the projectivization of the normal bundle of $\mathbf{V} \subset \mathbf{X}$

Lemma 5.2. *We assume* $r \leq 5$. *Let* $\rho' : \mathbf{X}' \dashrightarrow \mathbb{P}\left(\mathbf{S}_3 \otimes \overset{2}{\wedge} \mathbf{S}_1^\star\right)$ *be the rational map induced by* $\widetilde{\rho}$ *and denote by* $\mathbf{B}' \subset \mathbf{X}'$ *the indeterminacy scheme of* ρ'. *Then we have*

$$\mathbf{B}' = \mathbb{P}\left(\mathcal{N}_{\mathbf{V}/\mathbf{B}}\right) \subset \mathbb{P}\left(\mathcal{N}_{\mathbf{V}/\mathbf{X}}\right) = \mathbf{E}',$$

the projectivization of the normal bundle of $\mathbf{V}$ *in its thickenning* $\mathbf{B}$.

Proof. We look at the diagram of tangent/normal bundles over $\mathbf{V}$,

$$\begin{tikzcd}
T\mathbf{V} \arrow[r, equal] \arrow[d, tail] & T\mathbf{V} \arrow[d, tail] & \\
T\mathbf{B}_{|\mathbf{V}} \arrow[r, tail] \arrow[d, two heads] & T\mathbf{X}_{|\mathbf{V}} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{N}_{\mathbf{B}/\mathbf{X}|\mathbf{V}} \arrow[d, equal] \\
\mathcal{N}_{\mathbf{V}/\mathbf{B}} \arrow[r, tail] & \mathcal{N}_{\mathbf{V}/\mathbf{X}} \arrow[r, two heads] & \mathcal{N}_{\mathbf{B}/\mathbf{X}|\mathbf{V}}
\end{tikzcd} \tag{27}$$

which tells us that $\mathbb{P}\left(\mathcal{N}_{\mathbf{V}/\mathbf{B}}\right)$ embeds naturally into $\mathbf{E}' = \mathbb{P}\left(\mathcal{N}_{\mathbf{V}/\mathbf{X}}\right)$. Let $x' \in \mathbf{E}'$. Thus we may represent it as $x' = \lim_{\varepsilon \to 0}(L^2 + \varepsilon F', L^3 + \varepsilon G')$ for some $(F', G') \in T_{(L^2, L^3)}\mathbf{X}$ with nonzero image in $\mathcal{N}_{\mathbf{V}/\mathbf{X}}$. Here we think of $(L^2 + \varepsilon F', L^3 + \varepsilon G')$ as

a small arc in $\mathbf{X} \setminus \mathbf{V}$ for $\varepsilon \neq 0$. Hence it lifts to an arc in $\mathbf{X}' \setminus \mathbf{E}'$ which hits $x' \in \mathbf{E}'$ for $\varepsilon = 0$. As in (26) we find for $\varepsilon \neq 0$,

$$\rho(L^2+\varepsilon F', L^3+\varepsilon G') \begin{aligned}[t] &= \varepsilon LdL \wedge (2dG' - 3LdF') + \varepsilon^2 dF' \wedge dG' \\ &= LdL \wedge (2dG' - 3LdF') + \varepsilon dF' \wedge dG'. \end{aligned} \tag{28}$$

Now if x' is *not* in the indeterminacy locus, $\mathbf{B}'$, then we must have

$$\rho'(x') = \lim_{\varepsilon \to 0} \rho(L^2+\varepsilon F', L^3+\varepsilon G').$$

This limit is $\rho'(x') = LdL \wedge (2dG' - 3LdF')$ provided the expression is $\neq 0$. It is zero if and only if $G' = \frac{2}{3}LF'$, *i.e.*, x' is in $\mathbb{P}\left(\mathcal{N}_{\mathbf{V}/\mathbf{B}}\right)$. In this case, recalling (28),

$$\rho'(x') = dF' \wedge dG' = \frac{2}{3}F'dF' \wedge dL.$$

Since the right hand side must be (projectively) independent of representatives of $F' \in \mathbf{S}_2/\langle L^2\rangle$, we must have $dL \wedge dF' = 0$, a contradiction. Thus $LdL \wedge (2dG' - 3LdF')$ must be $\neq 0$, *i.e.*, x' is *not* in $\mathbb{P}\left(\mathcal{N}_{\mathbf{V}/\mathbf{B}}\right)$. This yields $\mathbb{P}\left(\mathcal{N}_{\mathbf{V}/\mathbf{B}}\right) \subseteq \mathbf{B}'$.

The dimension is given by

$$\dim \mathbf{B}' = \dim \mathbf{V} + \operatorname{rank} \mathcal{N}_{\mathbf{V}/\mathbf{B}} - 1 = r + \tbinom{r+2}{2} - 1 - r - 1 = N_2 - 1 = \tbinom{r+2}{2} - 2.$$

Thus we also have $\operatorname{codim} \mathbf{B}' = \operatorname{rank} \mathcal{N}_{\mathbf{B}'/\mathbf{X}'} = N_3 + 1$.

Unfortunately, for the other inclusion we don't know how to proceed coordinate-freewise. Using coordinates, with the help of computer algebra (SINGULAR), it can be checked (see [16]) that $\mathbf{B}'$ is smooth and of the right dimension $\dim \mathbb{P}\left(\mathcal{N}_{\mathbf{V}/\mathbf{B}}\right)$. This requires fixing r to low values, *e.g.,* $r \leq 5$. Here is an outline of the calculation for $r = 2$. We take affine coordinates $a_1, \ldots, a_5, b_1, \ldots, b_9$ for $\mathbb{P}(\mathbf{S}_2) \times \mathbb{P}(\mathbf{S}_3)$. Set

$$\begin{aligned} F &= x_0^2 + a_1x_0x_1 + a_2x_0x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2, \\ G &= x_0^3 + b_1x_0^2x_1 + b_2x_0^2x_2 + \cdots + b_8x_1x_2^2 + b_9x_2^3. \end{aligned}$$

We compute $dF \wedge dG$ expanding the 2×2 minors of the 2×3 matrix with rows the gradients of F, G. We find three cubics as coefficients of $dx_0 \wedge dx_1, dx_0 \wedge dx_2, dx_1 \wedge dx_2$. The indeterminacy locus, $\mathbf{B}$, is given by the ideal spanned by those thirty coefficients. Its jet of order one is spanned by nine independent linear equations, in agreement with the expected tangent space dimension, to wit, 5, the freedom of the quadric F. Continuing, we find next the local equations of the bi-Veronese, eliminating c_1, c_2 from the 5+9 equations obtained from the conditions

$$F = (x_0 + c_1x_1 + c_2x_2)^2,\ G = (x_0 + c_1x_1 + c_2x_2)^3.$$

We find that the ideal of the bi-Veronese is spanned by the 12 polynomials

$$\begin{gathered} 2b_1 - 3a_1,\ 4b_3 - 3a_1^2,\ 8b_6 - a_1^3,\ 2b_2 - 3a_2,\ 2b_4 - 3a_1a_2,\ 8b_7 - 3a_1^2a_2, \\ 4b_5 - 3a_2^2,\ 8b_8 - 3a_1a_2^2,\ 8b_9 - a_2^3,\ a_1^2 - 4a_3,\ a_1a_2 - 2a_4,\ a_2^2 - 4a_5. \end{gathered}$$

Accordingly, the blowup is covered by 12 affine patches, one for each choice of the principal generator for the exceptional ideal. The 9 generators involving a $b-$coefficient belong to the ideal of $\mathbf{B}$. It follows that the indeterminacy locus $\mathbf{B}'$ is disjoint from these nine neighborhoods. We are left with the 3 equations $4a_3 - a_1^2, 2a_4 - a_1a_2, 4a_5 - a_2^2$; these define the Veronese in $\mathbb{P}^5$. Choosing $\varepsilon = a_1^2 - 4a_3$

as the exceptional generator, the blowup is written as

$$\begin{cases} b_1 = \frac{1}{2}\varepsilon c_1 + \frac{3}{2}a_1,\ b_2 = \frac{1}{2}\varepsilon c_2 + \frac{3}{2}a_2, \\ b_3 = \frac{1}{4}\varepsilon c_3 + \frac{3}{4}a_1^2,\ b_4 = \frac{1}{2}\varepsilon c_4 + \frac{3}{2}a_1a_2, \\ b_5 = \frac{1}{4}\varepsilon c_5 + \frac{3}{4}a_2^2,\ b_6 = \frac{1}{8}\varepsilon c_6 + \frac{1}{8}a_1^3, \\ b_7 = \frac{1}{8}\varepsilon c_7 + \frac{3}{8}a_1^2a_2,\ b_8 = \frac{1}{8}\varepsilon c_8 + \frac{3}{8}a_1a_2^2, \\ b_9 = \frac{1}{8}\varepsilon c_9 + \frac{1}{8}a_2^3,\ a_4 = -\frac{1}{2}\varepsilon c_{10} + \frac{1}{2}a_1a_2, \\ a_5 = -\frac{1}{4}\varepsilon c_{11} + \frac{1}{4}a_2^2. \end{cases}$$

Substituting into the ideal of the indeterminacy locus, the original 30 generators become divisible by the local equation, ε, of the exceptional ideal. Dividing, we obtain the ideal of the indeterminacy locus upstairs, that is, of the induced rational map ρ' (cf. Lemma 5.2). We find the ideal of $\mathbf{B}'$ is presently generated by

$$\begin{gathered} c_1, c_2, 2c_4 + 3c_{10}, 2c_7 + 6c_{10}a_1 - 3a_2, 2c_5 + 3c_{11}, \\ 2c_8 + 6c_{10}a_2 + 3c_{11}a_1, 2c_9 + 3c_{11}a_2, 2c_3 - 3, 2c_6 - 3a_1, \underbrace{a_1^2 - 4a_3}_{\varepsilon}. \end{gathered}$$

Thus we see that the indeterminacy locus is contained in the exceptional divisor and we also learn that it is in fact a projective subbundle of the exceptional divisor $\mathbf{E}'$, in agreement with 5.2. □

Remark 5.2. Lemma 5.2 is valid only for small values of r, as stated and explained in the proof. But we conjecture that it is true for all r. It seems that a more conceptual proof is needed and probably it would involve some new idea.

At any rate, for each value of r, the validity of Lemma 5.2 is all we need to find the degree: We consider the following diagram displaying the resolution of the map $\widetilde{\rho} : X \dashrightarrow \mathbb{P}\left(\mathbf{S}_3 \otimes \overset{2}{\wedge}\mathbf{S}_1^\star\right)$.

$$\begin{tikzcd} \mathbf{E}'' \arrow[d] & & & \subset & \mathbf{X}'' \arrow[d] \arrow[ddr, "\rho''"] & \\ \mathbf{B}' & \subset & \mathbf{E}' \arrow[d] & \subset & \mathbf{X}' \arrow[d] \arrow[dr, dashed, "\widetilde{\rho}'"] & \\ & & \mathbf{V} & \subset & \mathbf{X} \arrow[r, dashed, "\widetilde{\rho}"] & \mathbb{P}\left(\mathbf{S}_3 \otimes \overset{2}{\wedge}\mathbf{S}_1^\star\right) \end{tikzcd}$$

The pullback of the hyperplane class via ρ'' can be written as

$$\rho''^{-1}\mathbf{h} = m_1\mathbf{h}_1 + m_2\mathbf{h}_2 + m_3\mathbf{e}' + m_4\mathbf{e}''$$

for suitable integers m_i, where we've denote the cycles $\mathbf{e}' = [\mathbf{E}'], \mathbf{e}'' = [\mathbf{E}'']$ and $\mathbf{h}_i$ the hyperplane class of each factor in $\mathbf{X} = \mathbb{P}(\mathbf{S}_2) \times \mathbb{P}(\mathbf{S}_3)$. The coefficients m_i will be determined using the Remark 5.1 and excision (cf. [5, 1.8, p. 21]). Over $\mathbf{U} = \mathbf{X} \backslash \mathbf{V}$ only $\mathbf{h}_1, \mathbf{h}_2$ survive and we have $\rho''^{-1}_{\mathbf{U}}\mathbf{h} = \widetilde{\rho}^{-1}_{\mathbf{U}}\mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2$ since $\rho_{\mathbf{U}}$ is defined by a bihomogeneous expression of bidegree 1,1. Put $\mathbf{U}' = \mathbf{X}' \setminus \mathbf{B}' = \mathbf{X}'' \setminus \mathbf{E}''$. The local calculations show that the image of $(\mathbf{S}_3 \otimes \overset{2}{\wedge}\mathbf{S}_1^\star)^\star \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S}_2)}(-1) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S}_3)}(-1) \to \mathcal{O}_{\mathbf{X}'}$ is equal to $\mathcal{O}_{\mathbf{X}'}(-\mathbf{E}') \cdot \mathcal{I}(\mathbf{B}')$. Blowing-up $\mathbf{B}'$, we find the surjection

$$(\mathbf{S}_3 \otimes \overset{2}{\wedge}\mathbf{S}_1^\star)^\star \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S}_2)}(-1) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S}_3)}(-1) \otimes \mathcal{O}_{\mathbf{X}''}(\mathbf{E}') \twoheadrightarrow \mathcal{O}_{\mathbf{X}''}(-\mathbf{E}'') = \mathcal{I}(\mathbf{B}')\mathcal{O}_{\mathbf{X}''}.$$

Thus, we have

$$(\widetilde{\rho})^{-1}\mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2 - \mathbf{e}' - \mathbf{e}''.$$

The degree is computed as

$$\int_{\mathbf{X}''} (\mathbf{h}_1 + \mathbf{h}_2 - \mathbf{e}' - \mathbf{e}'')^{N_2+N_3}.$$

Apart from the term $\int_{\mathbf{X}''} (\mathbf{h}_1 + \mathbf{h}_2)^{N_2+N_3} = \binom{N_2+N_3}{N_2}$, all others lie over $\mathbf{V}$. Since $\mathbf{h}_1 \cap \mathbf{V} = 2\mathbf{h}, \mathbf{h}_2 \cap \mathbf{V} = 3\mathbf{h}$ and $\mathbf{h}^{r+1} = 0$, we see that terms like $\mathbf{h}_1^i \mathbf{h}_1^j (\mathbf{e}')^k (\mathbf{e}'')^l$ give zero whenever $i + j > r$. Thus the relevant part of the integrand is

$$\sum_0^r \binom{N_2+N_3}{i} (5\mathbf{h})^i (-\mathbf{e}' - \mathbf{e}'')^{N_2+N_3-i}.$$

First we collect coefficients of $\mathbf{e}''$, then take the pushforward to $\mathbf{X}'$ using our knowledge of the normal bundle of $\mathbf{B}' \subset \mathbf{X}'$ and so on till $\mathbf{X}$. Thus

$$(\mathbf{e}'')^i = (\mathbf{e}'')^{i-1}\mathbf{e}'' \rightsquigarrow (-1)^{i-1} s_{i'}(\mathcal{N}_{\mathbf{B}'/\mathbf{X}'}) \cap [\mathbf{B}'],$$

with $i' = i - \operatorname{codim} \mathbf{B}' = i - N_3 - 1$. The sum above pushes forward to

$$\sum_0^r \tbinom{N_2+N_3}{i} (5\mathbf{h})^i (-1)^{N_2+N_3-i} \quad \Big(-(\mathbf{e}')^{N_2+N_3-i} + \sum_{j=\dim \mathbf{B}'}^{N_2+N_3-i} \tbinom{N_2+N_3-i}{j} (\mathbf{e}')^{N_2+N_3-i-j} (-1)^{j-1} s_{j'} \Big),$$

setting for short $s_{j'} = s_{j'}(\mathcal{N}_{\mathbf{B}'/\mathbf{X}'}) \cap [\mathbf{B}']$, with $j' = j - \operatorname{codim} \mathbf{B}' = j - N_3 - 1$. (Thus $s_{j'}$ is a cycle of dimension $N_2 - 1 - j' = N_2 + N_3 - j$.) These Segre classes can be derived from 5.2 as follows. We have the exact sequence

$$\mathcal{N}_{\mathbf{B}'/\mathbf{E}'} \rightarrowtail \mathcal{N}_{\mathbf{B}'/\mathbf{X}'} \twoheadrightarrow \mathcal{O}_{\mathbf{E}'}(\mathbf{E}')_{|\mathbf{B}'}. \tag{29}$$

We also recall that, for any exact sequence of vector bundles

$$\mathcal{E}' \rightarrowtail \mathcal{E} \twoheadrightarrow \mathcal{E}''$$

we have the formula for the normal bundle of $\mathbb{P}(\mathcal{E}') \subset \mathbb{P}(\mathcal{E})$

$$\mathcal{N}_{\mathbb{P}(\mathcal{E}')/\mathbb{P}(\mathcal{E})} = \mathcal{E}'' \otimes \mathcal{O}_{\mathcal{E}'}(1).$$

In view of (27), this yields

$$\mathcal{N}_{\mathbf{B}'/\mathbf{E}'} = \mathcal{N}_{\mathbf{B}/\mathbf{X}|\mathbf{V}} \otimes \mathcal{O}_{\mathcal{N}_{\mathbf{V}/\mathbf{B}}}(1).$$

The actual calculation is best performed using computer algebra. A script using Singular [8] is available at [16]. A sample of the first few values is listed below.

r	deg
2	770
3	6254612
4	481152797320
5	803161672838504856
6	36968358460592709286459400
7	53639021695280557844870264612516640
8	2759237622445467221610266591396121818496881016

(2,3)

As a final remark we mention that there is compelling computer algebra evidence indicating that the case of bidegree (2,3) carries over to the case $(2, 2m+1)$ with slight modifications. The indeterminacy locus of the rational map $\mathbf{X} = \mathbb{P}(\mathbf{S}_2) \times \mathbb{P}(\mathbf{S}_{2m+1}) \dashrightarrow \mathbb{P}(\mathbf{S}_{2m+1} \otimes \overset{2}{\wedge} \mathbf{S}_1^\star)$ given by $(F, G) \mapsto dF \wedge dG$ is again a thickening of the biveronese $\{(L^2, L^{2m+1}) \mid L \in \mathbb{P}(\mathbf{S}_1)\}$. Blowing up the reduced structure, the indeterminacy locus, $\mathbf{B}'$, of the induced rational map $\mathbf{X}' \dashrightarrow \mathbb{P}(\mathbf{S}_{2m+1} \otimes \overset{2}{\wedge} \mathbf{S}_1^\star)$ is no longer reduced for $m > 1$. Nevertheless, it still is a rather manageable complete intersection. In fact, we find local equations of $\mathbf{B}'$ of the form $e^m, f_1, ..., f_u$, with e denoting the equation of the exceptional divisor, and the f_i's define a projective subbundle of the exceptional divisor just as in the case (2,3).

$\deg \mathcal{R}(r, d_0 = 2, d_1 = 2m+1)$

d_1	deg $(\mathbb{P}^3)$	d_1	deg $(\mathbb{P}^4)$
5	27500627268	5	5858652068789831804
7	19062120397608	7	2734930355086609774678630
9	3910289698588916	9	118796991387599661786404269060
11	341013122932980120	11	9556673569317401629877052363742 00

Interpolating the first few values of odd d_1, we find for $\mathbb{P}^3$ the polynomial
$(t-1)\big(t^{26} + 55t^{25} + 1450t^{24} + 24616t^{23} + 305020t^{22} + 2961172t^{21} + 23561656t^{20} + 158392960t^{19} + 918866662t^{18} + 4670514826t^{17} + 21033417148t^{16} + 84615935632t^{15} + 305921226844t^{14} + 998318576836t^{13} + 2949392111320t^{12} + 7903552056256t^{11} + 19229223618721t^{10} + 41774679574903t^{9} + 72390849730794t^{8} + 15945324910344t^{7} - 541088235621216t^{6} - 2539188961011216t^{5} - 315410776482528t^{4} + 14933666207688192t^{3} + 85822791395378688t^{2} - 247712474710388736t + 162893498195312640\big)/3656994324480$.
It fits all values of $\deg \mathcal{R}(3, 2, t)$, $t = 2m+1$, up to $m = 35, d_1 = 71$, presently the physical limit of our computer's memory. It should be noted that $\deg \mathcal{R}(3, 2, 2t) = \binom{N_{2t}+N_2-1}{N_2} - \frac{2t}{2}\binom{N_{2t}+N_2-1}{N_2-1}$ is a polynomial in t of the same degree 27 as above.

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